

# DERIVED EQUIVALENCES FOR SYMMETRIC GROUPS AND $\mathfrak{sl}_2$ -CATEGORIFICATION

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**ABSTRACT.** We define and study  $\mathfrak{sl}_2$ -categorifications on abelian categories. We show in particular that there is a self-derived (even homotopy) equivalence categorifying the adjoint action of the simple reflection. We construct categorifications for blocks of symmetric groups and deduce that two blocks are splendidly Rickard equivalent whenever they have isomorphic defect groups and we show that this implies Broué's abelian defect group conjecture for symmetric groups. We give similar results for general linear groups over finite fields. The constructions extend to cyclotomic Hecke algebras. We also construct categorifications for category  $\mathcal{O}$  of  $\mathfrak{gl}_n(\mathbf{C})$  and for rational representations of general linear groups over  $\bar{\mathbf{F}}_p$ , where we deduce that two blocks corresponding to weights with the same stabilizer under the dot action of the affine Weyl group have equivalent derived (and homotopy) categories, as conjectured by Rickard.

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## 1. INTRODUCTION

The aim of this paper is to show that two blocks of symmetric groups with isomorphic defect groups have equivalent derived categories. We deduce in particular that Broué's abelian defect group conjecture holds for symmetric groups. We prove similar results for general linear groups over finite fields and cyclotomic Hecke algebras.

Recall that there is an action of  $\hat{\mathfrak{sl}}_p$  on the sum of Grothendieck groups of categories of  $k\mathfrak{S}_n$ -modules, for  $n \geq 0$ , where  $k$  is a field of characteristic  $p$ . The action of the generators  $e_i$  and  $f_i$  come from exact functors between modules ("i-induction" and "i-restriction"). The adjoint action of the simple reflections of the affine Weyl group can be categorified as functors between derived categories, following Rickard. The key point is to show that these functors are invertible, since two blocks have isomorphic defect groups if and only if they are in the same affine Weyl group orbit. This involves only an  $\mathfrak{sl}_2$ -action and we solve the problem in a more general framework.

We develop a notion of  $\mathfrak{sl}_2$ -categorification on an abelian category. This involves the data of adjoint exact functors  $E$  and  $F$  inducing an  $\mathfrak{sl}_2$ -action on the Grothendieck group and the data of endomorphisms  $X$  of  $E$  and  $T$  of  $E^2$  satisfying the defining relations of (degenerate) affine Hecke algebras.

Our main Theorem is a proof that the categorification  $\Theta$  of the simple reflection is a self-equivalence at the level of derived (and homotopy) categories. We achieve this in two steps. First, we show that there is a minimal categorification of string (=simple) modules coming from certain quotients of (degenerate) affine Hecke algebras : this reduces the proof of invertibility of  $\Theta$  to the case of the minimal categorification. There,  $\Theta$  becomes (up to shift) a self-equivalence of the abelian category.

Let us now describe in more detail the structure of this article. The first part §3 is devoted to the study of (degenerate) affine Hecke algebras of type  $A$  completed at a maximal ideal corresponding to a totally ramified central character. We construct (in §3.2) explicit decompositions of tensor products of ideals which we later translate into isomorphisms of functors. In §3.3, we introduce certain quotients, that turn out to be Morita equivalent to cohomology rings of Grassmannians. Part §4 recalls elementary results on adjunctions and on representations of  $\mathfrak{sl}_2$ .

Part §5 is devoted to the definition and study of  $\mathfrak{sl}_2$ -categorifications. We first define a weak version (§5.1), with functors  $E$  and  $F$  satisfying  $\mathfrak{sl}_2$ -relations in the Grothendieck group. This is enough to get filtrations of the category and to introduce a class of objects that control the abelian category. Then, in §5.2, we introduce the extra data of  $X$  and  $T$  which give the genuine  $\mathfrak{sl}_2$ -categorifications. This provides actions of (degenerate) affine Hecke algebras on powers of  $E$  and  $F$ . This leads immediately to two constructions of divided powers of  $E$  and

*F.* In order to study  $\mathfrak{sl}_2$ -categorifications, we introduce in §5.3 “minimal” categorifications of the simple  $\mathfrak{sl}_2$ -representations, based on the quotients introduced in §3.3. A key construction (§5.4.2) is a functor from such a minimal categorification to a given categorification, that allows to reduce part of the study of an arbitrary  $\mathfrak{sl}_2$ -categorification to this minimal case, where explicit computations can be carried out. This corresponds to the decomposition of the  $\mathfrak{sl}_2$ -representation on  $K_0$  into a direct sum of irreducible representations. We use this in §5.5 to prove a categorified version of the relation  $[e, f] = h$  and deduce a construction of categorifications on the module category of the endomorphism ring of “stable” objects in a given categorification.

Part §6 is devoted to the categorification of the simple reflection of the Weyl group. In §6.1, we construct a complex of functors categorifying this reflection, following Rickard. The main result is Theorem 6.4 in part §6.2, which shows that this complex induces a self-equivalence of the homotopy and of the derived category. The key step in the proof for the derived category is the case of a minimal categorification, where we show that the complex has homology concentrated in one degree (§6.3). The case of the homotopy category is reduced to the derived category thanks to the constructions of §5.5.

In part §7, we study various examples. We define (in §7.1)  $\mathfrak{sl}_2$ -categorifications on representations of symmetric groups and deduce derived and even splendid Rickard equivalences. We deduce a proof of Broué’s abelian defect group conjecture for blocks of symmetric groups. We give similar constructions for cyclotomic Hecke algebras (§7.2) and for general linear groups over a finite field in the non-defining characteristic case (§7.3) for which we also deduce the validity of Broué’s abelian defect group conjecture. We also construct  $\mathfrak{sl}_2$ -categorifications on category  $\mathcal{O}$  for  $\mathfrak{gl}_n$  (§7.4) and on rational representations of  $GL_n$  over an algebraically closed field of characteristic  $p > 0$  (§7.5). This answers in particular the GL case of a conjecture of Rickard on blocks corresponding to weights with same stabilizers under the dot action of the affine Weyl group. We also explain similar constructions for  $q$ -Schur algebras (§7.6) and provide morphisms of categorifications relating the previous constructions. A special role is played by the endomorphism  $X$ , which takes various incarnations : the Casimir in the rational representation case and the Jucys-Murphy elements in the Hecke algebra case. In the case of the general linear groups over a finite field, our construction seems to be new. Our last section (§7.7) provides various realizations of minimal categorifications, including one coming from the geometry of Grassmannian varieties.

Our general approach is inspired by [LLT], [Ar1], [Gr], and [GrVa] (cf [Rou3, §3.3]), and our strategy for proving the invertibility of  $\Theta$  is reminiscent of [DeLu, CaRi].

In a work in progress, we study the braid relations between the categorifications of the simple reflections, in the more general framework of categorifications of Kac-Moody algebras and in relation with Nakajima’s quiver variety constructions.

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## 2. NOTATIONS

Given an algebra  $A$ , we denote by  $A^{\text{opp}}$  the opposite algebra. We denote by  $A\text{-mod}$  the category of finitely generated  $A$ -modules. Given an abelian category  $\mathcal{A}$ , we denote by  $\mathcal{A}\text{-proj}$  the category of projective objects of  $\mathcal{A}$ .

Let  $\mathcal{C}$  be an additive category. We denote by  $\text{Comp}(\mathcal{C})$  the category of complexes of objects of  $\mathcal{C}$  and by  $K(\mathcal{C})$  the corresponding homotopy category.

Given an object  $M$  in an abelian category, we denote by  $\text{soc}(M)$  (resp.  $\text{hd}(M)$ ) the socle (resp. the head) of  $M$ , *i.e.*, the largest semi-simple subobject (resp. quotient) of  $M$ , when this exists.

We denote by  $K_0(\mathcal{A})$  the Grothendieck group of an exact category  $\mathcal{A}$ .

Given a functor  $F$ , we write sometimes  $F$  for the identity endomorphism  $\mathbf{1}_F$  of  $F$ .

### 3. AFFINE HECKE ALGEBRAS

**3.1. Definitions.** Let  $k$  be a field and  $q \in k^\times$ . We define a  $k$ -algebra  $H_n = H_n(q)$ .

**3.1.1. Non-degenerate case.** Assume  $q \neq 1$ . The affine Hecke algebra  $H_n(q)$  is the  $k$ -algebra with generators

$$T_1, \dots, T_{n-1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}$$

subject to the relations

$$\begin{aligned} (T_i + 1)(T_i - q) &= 0 \\ T_i T_j &= T_j T_i \quad (\text{when } |i - j| > 1) \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\ X_i X_j &= X_j X_i \\ X_i X_i^{-1} &= X_i^{-1} X_i = 1 \\ X_i T_j &= T_j X_i \quad (\text{when } i - j \neq 0, 1) \\ T_i X_i T_i &= q X_{i+1}. \end{aligned}$$

We denote by  $H_n^f(q)$  the subalgebra of  $H_n(q)$  generated by  $T_1, \dots, T_{n-1}$ . It is the Hecke algebra of the symmetric group  $\mathfrak{S}_n$ .

Let  $P_n = k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , a subalgebra of  $H_n(q)$  of Laurent polynomials. We put also  $P_{[i]} = k[X_i^{\pm 1}]$ .

**3.1.2. Degenerate case.** Assume  $q = 1$ . The degenerate affine Hecke algebra  $H_n(1)$  is the  $k$ -algebra with generators

$$T_1, \dots, T_{n-1}, X_1, \dots, X_n$$

subject to the relations

$$\begin{aligned} T_i^2 &= 1 \\ T_i T_j &= T_j T_i \quad (\text{when } |i - j| > 1) \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\ X_i X_j &= X_j X_i \\ X_i T_j &= T_j X_i \quad (\text{when } i - j \neq 0, 1) \\ X_{i+1} T_i &= T_i X_i + 1. \end{aligned}$$

Note that the degenerate affine Hecke algebra is not the specialization of the affine Hecke algebra.

We put  $P_n = k[X_1, \dots, X_n]$ , a polynomial subalgebra of  $H_n(1)$ . We put also  $P_{[i]} = k[X_i]$ . The subalgebra  $H_n^f(1)$  of  $H_n(1)$  generated by  $T_1, \dots, T_{n-1}$  is the group algebra  $k\mathfrak{S}_n$  of the symmetric group.

3.1.3. We put  $H_n = H_n(q)$  and  $H_n^f = H_n^f(q)$ . There is an isomorphism  $H_n \xrightarrow{\sim} H_n^{\text{opp}}, T_i \mapsto T_i, X_i \mapsto X_i$ . It allows us to switch between right and left  $H_n$ -modules. There is an automorphism of  $H_n$  defined by  $T_i \mapsto T_{n-i}, X_i \mapsto \tilde{X}_{n-i+1}$ , where  $\tilde{X}_i = X_i^{-1}$  if  $q \neq 1$  and  $\tilde{X}_i = -X_i$  if  $q = 1$ .

We denote by  $l : \mathfrak{S}_n \rightarrow \mathbf{N}$  the length function. We put  $s_i = (i, i+1) \in \mathfrak{S}_n$ . Given  $w = s_{i_1} \cdots s_{i_r}$  a reduced decomposition of an element  $w \in \mathfrak{S}_n$  (i.e.,  $r = l(w)$ ), we put  $T_w = T_{s_{i_1}} \cdots T_{s_{i_r}}$ .

We have  $H_n = H_n^f \otimes P_n = P_n \otimes H_n^f$ .

We have an action of  $\mathfrak{S}_n$  on  $P_n$  by permutation of the variables. Given  $p \in P_n$ , we have [Lu, Proposition 3.6]

$$(1) \quad T_i p - s_i(p) T_i = \begin{cases} (q-1)(1 - X_i X_{i+1}^{-1})^{-1}(p - s_i(p)) & \text{if } q \neq 1 \\ (X_{i+1} - X_i)^{-1}(p - s_i(p)) & \text{if } q = 1 \end{cases}$$

Note that  $(P_n)^{\mathfrak{S}_n} \subset Z(H_n)$  (this is actually an equality, a result of Bernstein).

3.1.4. Let  $1$  (resp.  $\text{sgn}$ ) be the one-dimensional representation of  $H_n^f$  given by  $T_{s_i} \mapsto q$  (resp.  $T_{s_i} \mapsto -1$ ). Let  $\tau \in \{1, \text{sgn}\}$ . We put

$$c_n^\tau = \sum_{w \in \mathfrak{S}_n} q^{-l(w)} \tau(T_w) T_w.$$

We have  $c_n^\tau \in Z(H_n^f)$ . We have  $c_n^1 = \sum_{w \in \mathfrak{S}_n} T_w$  and  $c_n^{\text{sgn}} = \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} T_w$ , and  $c_n^1 c_n^{\text{sgn}} = c_n^{\text{sgn}} c_n^1 = 0$  for  $n \geq 2$ .

More generally, given  $1 \leq i \leq j \leq n$ , we denote by  $\mathfrak{S}_{[i,j]}$  the symmetric group on  $[i, j] = \{i, i+1, \dots, j\}$ , we define similarly  $H_{[i,j]}^f$ ,  $H_{[i,j]}$  and we put  $c_{[i,j]}^\tau = \sum_{w \in \mathfrak{S}_{[i,j]}} q^{-l(w)} \tau(T_w) T_w$ .

Given  $I$  a subset of  $\mathfrak{S}_n$  we put  $c_I^\tau = \sum_{w \in I} q^{-l(w)} \tau(T_w) T_w$ . We have

$$c_n^\tau = c_{[\mathfrak{S}_n/\mathfrak{S}_i]}^\tau c_i^\tau = c_i^\tau c_{[\mathfrak{S}_i \setminus \mathfrak{S}_n]}^\tau$$

where  $[\mathfrak{S}_n/\mathfrak{S}_i]$  (resp.  $[\mathfrak{S}_i \setminus \mathfrak{S}_n]$ ) is the set of minimal length representatives of right (resp. left) cosets.

Given  $M$  a projective  $H_n^f$ -module, then  $c_n^\tau M = \{m \in M \mid hm = \tau(h)m \text{ for all } h \in H_n^f\}$  and the multiplication map  $c_n^\tau H_n^f \otimes_{H_n^f} M \xrightarrow{\sim} c_n^\tau M$  is an isomorphism. Given  $N$  an  $H_n$ -module, then the canonical map  $c_n^\tau H_n^f \otimes_{H_n^f} N \xrightarrow{\sim} c_n^\tau H_n \otimes_{H_n} N$  is an isomorphism.

**3.2. Totally ramified central character.** We gather here a number of properties of (degenerate) affine Hecke algebras after completion at a maximally ramified central character. Compared to classical results, some extra complications arise from the possibility of  $n!$  being 0 in  $k$ .

3.2.1. We fix  $a \in k$ , with  $a \neq 0$  if  $q \neq 1$ . We put  $x_i = X_i - a$ . Let  $\mathfrak{m}_n$  be the maximal ideal of  $P_n$  generated by  $x_1, \dots, x_n$  and let  $\mathfrak{n}_n = (\mathfrak{m}_n)^{\mathfrak{S}_n}$ .

Let  $e_m(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_m \leq n} x_{i_1} \cdots x_{i_m} \in P_n^{\mathfrak{S}_n}$  be the  $m$ -th elementary symmetric function. Then,  $x_n^n = \sum_{i=0}^{n-1} (-1)^{n+i+1} x_n^i e_{n-i}(x_1, \dots, x_n)$ . So,  $x_n^l \in \bigoplus_{i=0}^{n-1} x_n^i \mathfrak{n}_n$  for  $l \geq n$ . Via Galois theory, we deduce that  $P_n^{\mathfrak{S}_{n-1}} = \bigoplus_{i=0}^{n-1} x_n^i P_n^{\mathfrak{S}_n}$ . Using that the multiplication map  $P_j^{\mathfrak{S}_j} \otimes P_{[j+1, n]} \xrightarrow{\sim} P_n^{\mathfrak{S}_j}$  is an isomorphism, we deduce by induction that

$$(2) \quad P_n^{\mathfrak{S}_r} = \bigoplus_{0 \leq a_i < r+i} x_{r+1}^{a_1} \cdots x_n^{a_{n-r}} P_n^{\mathfrak{S}_n}.$$

3.2.2. We denote by  $\widehat{P_n^{\mathfrak{S}_n}}$  the completion of  $P_n^{\mathfrak{S}_n}$  at  $\mathfrak{n}_n$ , and we put  $\hat{P}_n = P_n \otimes_{P_n^{\mathfrak{S}_n}} \widehat{P_n^{\mathfrak{S}_n}}$  and  $\hat{H}_n = H_n \otimes_{P_n^{\mathfrak{S}_n}} \widehat{P_n^{\mathfrak{S}_n}}$ . The canonical map  $\hat{P}_n^{\mathfrak{S}_n} \xrightarrow{\sim} \widehat{P_n^{\mathfrak{S}_n}}$  is an isomorphism, since  $\widehat{P_n^{\mathfrak{S}_n}}$  is flat over  $P_n^{\mathfrak{S}_n}$ .

We denote by  $\mathcal{N}_n$  the category of locally nilpotent  $\hat{H}_n$ -modules, *i.e.*, the category of  $H_n$ -modules on which  $\mathfrak{n}_n$  acts locally nilpotently : an  $H_n$ -module  $M$  is in  $\mathcal{N}_n$  if for every  $m \in M$ , there is  $i > 0$  such that  $\mathfrak{n}_n^i m = 0$ .

We put  $\bar{H}_n = H_n / (H_n \mathfrak{n}_n)$  and  $\bar{P}_n = P_n / (P_n \mathfrak{n}_n)$ . The multiplication gives an isomorphism  $\bar{P}_n \otimes H_n^f \xrightarrow{\sim} \bar{H}_n$ . The canonical map

$$\bigoplus_{0 \leq a_i < i} kx_1^{a_1} \cdots x_n^{a_n} \xrightarrow{\sim} \bar{P}_n$$

is an isomorphism, hence  $\dim_k \bar{H}_n = (n!)^2$ .

The unique simple object of  $\mathcal{N}_n$  is [Ka, Theorem 2.2]

$$K_n = H_n \otimes_{P_n} P_n / \mathfrak{m}_n \simeq \bar{H}_n c_n^\tau.$$

It has dimension  $n!$  over  $k$ . It follows that the canonical surjective map  $\bar{H}_n \rightarrow \text{End}_k(K_n)$  is an isomorphism, hence  $\bar{H}_n$  is a simple split  $k$ -algebra.

Since  $K_n$  is a free module over  $H_n^f$ , it follows that any object of  $\mathcal{N}_n$  is free by restriction to  $H_n^f$ . From §3.1.4, we deduce that for any  $M \in \mathcal{N}_n$ , the canonical map  $c_n^\tau H_n \otimes_{H_n} M \xrightarrow{\sim} c_n^\tau M$  is an isomorphism.

**Remark 3.1.** We have excluded the case of the affine Weyl group algebra (the affine Hecke algebra at  $q = 1$ ). Indeed, in that case  $K_n$  is not simple (when  $n \geq 2$ ) and  $\bar{H}_n$  is not a simple algebra. When  $n = 2$ , we have  $\bar{H}_n \simeq (k[x]/(x^2)) \rtimes \mu_2$ , where the group  $\mu_2 = \{\pm 1\}$  acts on  $x$  by multiplication.

3.2.3. Let  $f : M \rightarrow N$  be a morphism of finitely generated  $\hat{P}_n^{\mathfrak{S}_n}$ -modules. Then,  $f$  is surjective if and only if  $f \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n^{\mathfrak{S}_n} / \hat{\mathfrak{n}}_n$  is surjective.

**Lemma 3.2.** *We have isomorphisms*

$$\hat{H}_n c_n^\tau \otimes_k \bigoplus_{i=0}^{n-1} x_n^i k \xrightarrow[\sim]{\text{can}} \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n^{\mathfrak{S}_{n-1}} \xrightarrow[\sim]{\text{mult}} \hat{H}_n c_{n-1}^\tau.$$

*Proof.* The first isomorphism follows from the decomposition of  $\hat{P}_n^{\mathfrak{S}_{n-1}}$  in (2).

Let us now study the second map. Note that both terms are free  $\hat{P}_n^{\mathfrak{S}_n}$ -modules of rank  $n \cdot n!$ , since  $\hat{H}_n c_{n-1}^\tau \simeq \hat{P}_n \otimes H_n^f c_{n-1}^\tau$ . Consequently, it suffices to show that the map is surjective. Thanks to the remark above, it is enough to check surjectivity after applying  $-\otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n^{\mathfrak{S}_n}/\mathfrak{n}_n$ .

Note that the canonical surjective map  $k[x_n] \rightarrow P_n^{\mathfrak{S}_{n-1}} \otimes_{P_n^{\mathfrak{S}_n}} P_n^{\mathfrak{S}_n}/\mathfrak{n}_n$  factors through  $k[x_n]/(x_n^n)$  (cf §3.2.1). So, we have to show that the multiplication map  $f : \bar{H}_n c_n^\tau \otimes k[x_n]/(x_n^n) \rightarrow \bar{H}_n c_{n-1}^\tau$  is surjective. This is a morphism of  $(\bar{H}_n, k[x_n]/(x_n^n))$ -bimodules. The elements  $c_n^\tau, c_n^\tau x_n, \dots, c_n^\tau x_n^{n-1}$  of  $\bar{H}_n$  are linearly independent, hence the image of  $f$  is a faithful  $(k[x_n]/(x_n^n))$ -module. It follows that  $f$  is injective, since  $\bar{H}_n c_n^\tau$  is a simple  $\bar{H}_n$ -module. Now,  $\dim_k \bar{H}_n c_{n-1}^\tau = n \cdot n!$ , hence  $f$  is an isomorphism.  $\square$

Let  $M$  be a  $k\mathfrak{S}_n$ -module. We put  $\Lambda^{\mathfrak{S}_n} M = M/(\sum_{0 < i < n} M^{s_i})$ . If  $n! \in k^\times$ , then  $\Lambda^{\mathfrak{S}_n} M$  is the largest quotient of  $M$  on which  $\mathfrak{S}_n$  acts via the sign character. Note that given a vector space  $V$ , then  $\Lambda^{\mathfrak{S}_n}(V^{\otimes n}) = \Lambda^n V$ .

**Proposition 3.3.** *Let  $\{\tau, \tau'\} = \{1, \text{sgn}\}$  and  $r \leq n$ . We have isomorphisms*

$$\hat{H}_n c_n^\tau \otimes_k \bigoplus_{0 \leq a_i < n-r+i} x_{n-r+1}^{a_1} \cdots x_n^{a_r} k \xrightarrow[\sim]{\text{can}} \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n^{\mathfrak{S}_{[1, n-r]}} \xrightarrow[\sim]{\text{mult}} \hat{H}_n c_{[1, n-r]}^\tau.$$

There is a commutative diagram

$$\begin{array}{ccc} \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n^{\mathfrak{S}_{[1, n-r]}} & & \\ \text{can} \downarrow & \searrow^{x \otimes y \mapsto xy c_{[n-r+1, n]}^{\tau'}} & \\ \hat{H}_n c_n^\tau \otimes_k \bigoplus_{0 \leq a_1 < \dots < a_r < n} x_{n-r+1}^{a_1} \cdots x_n^{a_r} k & \xrightarrow[\sim]{\text{can}} \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \Lambda^{\mathfrak{S}_{[n-r+1, n]}} \hat{P}_n^{\mathfrak{S}_{[1, n-r]}} & \cdots \xrightarrow{\sim} \hat{H}_n c_{[1, n-r]}^\tau c_{[n-r+1, n]}^{\tau'} \end{array}$$

*Proof.* The multiplication map  $H_n \otimes_{H_{n-i}} H_{n-i} c_{n-i}^\tau \rightarrow H_n c_{n-i}^\tau$  is an isomorphism (cf §3.1.4). It follows from Lemma 3.2 that multiplication is an isomorphism

$$\hat{H}_n c_{n-r+1}^\tau \otimes \bigoplus_{i=0}^{n-r} x_{n-r+1}^i k \xrightarrow{\sim} \hat{H}_n c_{n-r}^\tau$$

and the first statement follows by descending induction on  $r$ .

The surjectivity of the diagonal map follows from the first statement of the Proposition.

Let  $p \in \hat{P}_n^{s_i}$ . Then,  $c_{[i, i+1]}^1 p = p c_{[i, i+1]}^1$ . It follows that  $c_{[i, i+1]}^\tau p c_{[i, i+1]}^{\tau'} = 0$ , hence  $c_n^\tau p c_{[n-r+1, n]}^{\tau'} = 0$  whenever  $i \geq n-r+1$ . This shows the factorization property (existence of the dotted arrow).

Note that  $\Lambda^{\mathfrak{S}_{[n-r+1, n]}} \hat{P}_n^{\mathfrak{S}_{n-r}}$  is generated by  $\bigoplus_{0 \leq a_1 < \dots < a_r < n} x_{n-r+1}^{a_1} \cdots x_n^{a_r} k$  as a  $\hat{P}_n^{\mathfrak{S}_n}$ -module (cf (2)). It follows that we have surjective maps

$$\hat{H}_n c_n^\tau \otimes_k \bigoplus_{0 \leq a_1 < \dots < a_r < n} x_{n-r+1}^{a_1} \cdots x_n^{a_r} k \twoheadrightarrow \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \Lambda^{\mathfrak{S}_{[n-r+1, n]}} \hat{P}_n^{\mathfrak{S}_{n-r}} \twoheadrightarrow \hat{H}_n c_{n-r}^\tau c_{[n-r+1, n]}^{\tau'}.$$

Now the first and last terms above are free  $\hat{P}_n$ -modules of rank  $\binom{n}{r}$ , hence the maps are isomorphisms.  $\square$

**Lemma 3.4.** *Let  $r \leq n$ . We have  $c_r^\tau \hat{H}_n c_n^\tau = \hat{P}_n^{\mathfrak{S}_r} c_n^\tau$ ,  $c_n^\tau \hat{H}_n c_r^\tau = c_n^\tau \hat{P}_n^{\mathfrak{S}_r}$  and the multiplication maps  $c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_r^\tau \xrightarrow{\sim} c_n^\tau \hat{H}_n c_r^\tau$  and  $c_r^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_n^\tau \xrightarrow{\sim} c_r^\tau \hat{H}_n c_n^\tau$  are isomorphisms.*

*Proof.* We have an isomorphism  $\hat{P}_n \xrightarrow{\sim} \hat{H}_n c_n^\tau$ ,  $p \mapsto pc_n^\tau$ . Let  $h \in \hat{H}_n$ . We have  $c_n^\tau h c_n^\tau = pc_n^\tau$  for some  $p \in \hat{P}_n$ . Since  $T_i c_n^\tau = \tau(T_i) c_n^\tau$ , it follows that  $T_i p c_n^\tau = \tau(T_i) p c_n^\tau$ . So,  $(T_i p - s_i(p) T_i) c_n^\tau = \tau(T_i)(p - s_i(p)) c_n^\tau$ , hence  $p - s_i(p) = 0$ , using the formula (1). It follows that  $c_n^\tau \hat{H}_n c_n^\tau \subseteq \hat{P}_n^{\mathfrak{S}_n} c_n^\tau$ .

By Proposition 3.3, the multiplication map  $\hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n \xrightarrow{\sim} \hat{H}_n$  is an isomorphism. So, the multiplication map  $c_n^\tau \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n \xrightarrow{\sim} c_n^\tau \hat{H}_n$  is an isomorphism, hence the canonical map  $c_n^\tau \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n \xrightarrow{\sim} \hat{P}_n^{\mathfrak{S}_n} c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n$  is an isomorphism. We deduce that  $c_n^\tau \hat{H}_n c_n^\tau = \hat{P}_n^{\mathfrak{S}_n} c_n^\tau$ .

Similarly (replacing  $n$  by  $r$  above), we have  $c_r^\tau \hat{P}_r^{\mathfrak{S}_r} c_r^\tau = c_r^\tau \hat{P}_r^{\mathfrak{S}_r}$ . Since  $P_n^{\mathfrak{S}_r} = P_r^{\mathfrak{S}_r} P_{[r+1, n]}$  (cf §3.2.1), we deduce that

$$c_n^\tau \hat{H}_n c_r^\tau = c_n^\tau \hat{P}_n c_r^\tau = c_n^\tau \hat{P}_r c_r^\tau \hat{P}_{[r+1, n]} = c_n^\tau \hat{P}_r^{\mathfrak{S}_r} \hat{P}_{[r+1, n]} = c_n^\tau \hat{P}_n^{\mathfrak{S}_r}.$$

By Proposition 3.3,  $c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_r^\tau$  is a free  $\hat{P}_n^{\mathfrak{S}_r}$ -module of rank 1. So, the multiplication map  $c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_r^\tau \rightarrow c_n^\tau \hat{H}_n c_r^\tau$  is a surjective morphism between free  $\hat{P}_n^{\mathfrak{S}_r}$ -modules of rank 1, hence it is an isomorphism.

The cases where  $c_r^\tau$  is on the left are similar.  $\square$

**Proposition 3.5.** *The functors  $H_n c_n^\tau \otimes_{P_n^{\mathfrak{S}_n}} -$  and  $c_n^\tau H_n \otimes_{H_n} -$  are inverse equivalences of categories between the category of  $P_n^{\mathfrak{S}_n}$ -modules that are locally nilpotent for  $\mathfrak{n}_n$  and  $\mathcal{N}_n$ .*

*Proof.* By Proposition 3.3, the multiplication map  $\hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n \xrightarrow{\sim} \hat{H}_n$  is an isomorphism. It follows that the morphism of  $(\hat{H}_n, \hat{H}_n)$ -bimodules

$$\hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} c_n^\tau \hat{H}_n \xrightarrow{\sim} \hat{H}_n, \quad hc \otimes ch' \mapsto hch'$$

is an isomorphism.

Since  $\hat{P}_n^{\mathfrak{S}_n}$  is commutative, it follows from Lemma 3.4 that the  $(\hat{P}_n^{\mathfrak{S}_n}, \hat{P}_n^{\mathfrak{S}_n})$ -bimodules  $\hat{P}_n^{\mathfrak{S}_n}$  and  $c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_n^\tau$  are isomorphic.  $\square$

### 3.3. Quotients.

3.3.1. We denote by  $\bar{H}_{i, n}$  the image of  $H_i$  in  $\bar{H}_n$  for  $0 \leq i \leq n$ . Let  $\bar{P}_{i, n} = P_i / (P_i \cap (P_n \mathfrak{n}_n))$ . We have an isomorphism  $H_i^f \otimes \bar{P}_{i, n} \xrightarrow[\sim]{\text{mult}} \bar{H}_{i, n}$ .

Since  $P_n^{\mathfrak{S}_{[i+1, n]}} = \bigoplus_{0 \leq a_l \leq n-l} x_1^{a_1} \cdots x_i^{a_i} P_n^{\mathfrak{S}_n}$  (cf (2)), we deduce that  $P_i = \bigoplus_{0 \leq a_l \leq n-l} x_1^{a_1} \cdots x_i^{a_i} \oplus (\mathfrak{n}_n P_i \cap P_i)$  and  $\mathfrak{n}_n P_i \cap P_i = \mathfrak{n}_n P_n \cap P_i$ , hence the canonical map

$$(3) \quad \bigoplus_{0 \leq a_l \leq n-l} x_1^{a_1} \cdots x_i^{a_i} \xrightarrow{\sim} \bar{P}_{i, n}$$

is an isomorphism. We will identify such a monomial  $x_1^{a_1} \cdots x_i^{a_i}$  with its image in  $\bar{P}_{i, n}$ . Note that  $\dim_k \bar{P}_{i, n} = \frac{n!}{(n-i)!}$ .

The kernel of the action of  $P_i^{\mathfrak{S}_i}$  by right multiplication on  $\bar{H}_{i, n} c_i^\tau$  is  $P_i^{\mathfrak{S}_i} \cap \mathfrak{n}_n P_n$ . By Proposition 3.5, we have a Morita equivalence between  $\bar{H}_{i, n}$  and  $Z_{i, n} = P_i^{\mathfrak{S}_i} / (P_i^{\mathfrak{S}_i} \cap \mathfrak{n}_n P_n)$ . Note that  $\bar{H}_{i, n} c_i^\tau$  is the unique indecomposable projective  $\bar{H}_{i, n}$ -module and  $\dim_k \bar{H}_{i, n} = i! \dim_k \bar{H}_{i, n} c_i^\tau$ . So,  $\dim_k Z_{i, n} = \frac{1}{(i!)^2} \dim_k \bar{H}_{i, n} = \binom{n}{i}$  and  $Z_{i, n} = Z(\bar{H}_{i, n})$ .



We denote by  $P(r, s)$  the set of partitions  $\mu = (\mu_1 \geq \dots \geq \mu_r \geq 0)$  with  $\mu_1 \leq s$ . Given  $\mu \in P(r, s)$ , we denote by  $m_\mu$  the corresponding monomial symmetric function

$$m_\mu(x_1, \dots, x_r) = \sum_{\sigma} x_1^{\mu_{\sigma(1)}} \dots x_r^{\mu_{\sigma(r)}}$$

where  $\sigma$  runs over left coset representatives of  $\mathfrak{S}_r$  modulo the stabilizer of  $(\mu_1, \dots, \mu_r)$ .

The isomorphism (3) shows that the canonical map from  $\bigoplus_{\mu \in P(i, n-i)} km_\mu(x_1, \dots, x_i)$  to  $\bar{P}_{i,n}$  is injective, with image contained in  $Z_{i,n}$ . Comparing dimensions, it follows that the canonical map

$$\bigoplus_{\mu \in P(i, n-i)} km_\mu(x_1, \dots, x_i) \xrightarrow{\sim} Z_{i,n}$$

is an isomorphism.

Also, comparing dimensions, one sees that the canonical surjective maps

$$P_i \otimes_{P_i^{\mathfrak{S}_i}} Z_{i,n} \xrightarrow{\sim} \bar{P}_{i,n} \text{ and } H_i \otimes_{P_i^{\mathfrak{S}_i}} Z_{i,n} \xrightarrow{\sim} \bar{H}_{i,n}$$

are isomorphisms.

**3.3.2.** Let  $G_{i,n}$  be the Grassmannian variety of  $i$ -dimensional subspaces of  $\mathbf{C}^n$  and  $G_n$  be the variety of complete flags in  $\mathbf{C}^n$ . The canonical morphism  $p : G_n \rightarrow G_{i,n}$  induces an injective morphism of algebras  $p^* : H^*(G_{i,n}) \rightarrow H^*(G_n)$  (cohomology is taken with coefficients in  $k$ ). We identify  $G_n$  with  $GL_n/B$ , where  $B$  is the stabilizer of the standard flag  $(\mathbf{C}(1, 0, \dots, 0) \subset \dots \subset \mathbf{C}^n)$ . Let  $L_j$  be the line bundle associated to the character of  $B$  given by the  $j$ -th diagonal coefficient. We have an isomorphism  $\bar{P}_n \xrightarrow{\sim} H^*(G_n)$  sending  $x_j$  to the first Chern class of  $L_j$ . It multiplies degrees by 2. Now,  $p^*H^*(G_{i,n})$  coincides with the image of  $P_i^{\mathfrak{S}_i}$  in  $\bar{P}_n$ . So, we have obtained an isomorphism

$$Z_{i,n} \xrightarrow{\sim} H^*(G_{i,n}).$$

Since  $G_{i,n}$  is projective, smooth and connected of dimension  $i(n-i)$ , Poincaré duality says that the cup product  $H^j(G_{i,n}) \times H^{2i(n-i)-j}(G_{i,n}) \rightarrow H^{2i(n-i)}(G_{i,n})$  is a perfect pairing. Via the isomorphism  $H^{2i(n-i)}(G_{i,n}) \xrightarrow{\sim} k$  given by the fundamental class, this provides  $H^*(G_{i,n})$  with a structure of a symmetric algebra.

Note that the algebra  $\bar{H}_{i,n}$  is isomorphic to the ring of  $i! \times i!$  matrices over  $H^*(G_{i,n})$  and it is a symmetric algebra. Up to isomorphism, it is independent of  $a$  and  $q$ .

**3.3.3.** Let  $i \leq j$ . We have

$$\bar{H}_{j,n} = \bar{H}_{i,n} \otimes \bigoplus_{\substack{w \in [\mathfrak{S}_i \setminus \mathfrak{S}_j] \\ 0 \leq a_l \leq n-l}} kx_{i+1}^{a_{i+1}} \dots x_j^{a_j} \otimes kT_w$$

hence  $\bar{H}_{j,n}$  is a free  $\bar{H}_{i,n}$ -module of rank  $\frac{(n-i)!j!}{(n-j)!i!}$ .

**Lemma 3.6.** *The  $H_i$ -module  $c_{[i+1,n]}^\tau K_n$  has a simple socle and head.*

*Proof.* By Proposition 3.3, multiplication gives an isomorphism

$$\bigoplus_{0 \leq a_l < l} x_{i+1}^{a_{i+1}} \dots x_n^{a_n} \otimes c_{[i+1,n]}^\tau H_{[i+1,n]} \xrightarrow{\sim} H_{[i+1,n]},$$

hence gives an isomorphism of  $\bar{H}_{i,n}$ -modules

$$\bigoplus_{0 \leq a_i < l} x_{i+1}^{a_1} \cdots x_n^{a_{n-i}} \otimes c_{[i+1,n]}^\tau \bar{H}_n \xrightarrow{\sim} \bar{H}_n.$$

Since  $\bar{H}_n$  is a free  $\bar{H}_{i,n}$ -module of rank  $\frac{(n-i)!n!}{i!}$ , it follows that hence  $c_{[i+1,n]}^\tau \bar{H}_n$  is a free  $\bar{H}_{i,n}$ -module of rank  $\frac{n!}{i!}$ . We have  $\bar{H}_{i,n} \simeq i! \cdot M$  as  $\bar{H}_{i,n}$ -modules, where  $M$  has a simple socle and head. Since in addition  $\bar{H}_n \simeq n! \cdot K_n$  as  $\bar{H}_n$ -modules, we deduce that  $c_{[i+1,n]}^\tau K_n \simeq M$  has a simple socle and head.  $\square$

**Lemma 3.7.** *Let  $r \leq l \leq n$ . We have isomorphisms*

$$\begin{array}{ccc} \bigoplus_{0 \leq a_i \leq n-i} x_1^{a_1} \cdots x_{l-r}^{a_{l-r}} k \otimes \bigoplus_{\mu \in P(r,n-l)} m_\mu(x_{l-r+1}, \dots, x_l) k & \xrightarrow[a \otimes b \mapsto abc_l^\tau]{\sim} & c_{[l-r+1,l]}^\tau \bar{H}_{l,n} c_l^\tau \\ \downarrow \sim a \otimes b \mapsto ac_n^\tau \otimes b & & \uparrow \sim \text{mult} \\ \bar{H}_{l-r,n} c_{l-r}^\tau \otimes \bigoplus_{\mu \in P(r,n-l)} m_\mu(x_{l-r+1}, \dots, x_l) k & & c_{[l-r+1,l]}^\tau \bar{H}_{l,n} \otimes_{\bar{H}_{l,n}} \bar{H}_{l,n} c_l^\tau \end{array}$$

*Proof.* Let  $L = \bigoplus_{\mu \in P(r,n-l), 0 \leq a_i \leq n-i} m_\mu(x_{l-r+1}, \dots, x_l) x_1^{a_1} \cdots x_{l-r}^{a_{l-r}} k$ .

We have  $L \cap \mathfrak{n}_n P_n = 0$  (cf (3)), hence the canonical map  $f : L \rightarrow P_l^{\mathfrak{S}_{[l-r+1,l]}} \otimes_{P_l^{\mathfrak{S}_l}} Z_{l,n}$  is injective. Since  $\dim_k Z_{l,n} = \binom{n}{l}$  and  $P_l^{\mathfrak{S}_{[l-r+1,l]}}$  is a free  $P_l^{\mathfrak{S}_l}$ -module of rank  $\frac{l!}{r!}$ , it follows that  $f$  is an isomorphism. Now, we have an isomorphism (Lemma 3.4)

$$\hat{P}_l^{\mathfrak{S}_{[l-r+1,l]}} \xrightarrow{\sim} c_{[l-r+1,l]}^\tau \hat{H}_l c_l^\tau, \quad a \mapsto ac_l^\tau.$$

Consequently, the horizontal map of the Lemma is an isomorphism.

As seen in §3.3.1, the left vertical map is an isomorphism. By Lemma 3.4, the right vertical map is also an isomorphism.  $\square$

## 4. REMINDERS

### 4.1. Adjunctions.

4.1.1. Let  $\mathcal{C}$  and  $\mathcal{C}'$  be two categories. Let  $(G, G^\vee)$  be an adjoint pair of functors,  $G : \mathcal{C} \rightarrow \mathcal{C}'$  and  $G^\vee : \mathcal{C}' \rightarrow \mathcal{C}$ : this is the data of two morphisms  $\eta : \text{Id}_{\mathcal{C}} \rightarrow G^\vee G$  (the unit) and  $\varepsilon : GG^\vee \rightarrow \text{Id}_{\mathcal{C}'}$  (the counit), such that  $(\varepsilon \mathbf{1}_G) \circ (\mathbf{1}_{G^\vee} \eta) = \mathbf{1}_G$  and  $(\mathbf{1}_{G^\vee} \varepsilon) \circ (\eta \mathbf{1}_{G^\vee}) = \mathbf{1}_{G^\vee}$ . We have then a canonical isomorphism functorial in  $X \in \mathcal{C}$  and  $X' \in \mathcal{C}'$

$$\gamma_G(X, X') : \text{Hom}(GX, X') \xrightarrow{\sim} \text{Hom}(X, G^\vee X'), \quad f \mapsto G^\vee(f) \circ \eta(X), \quad \varepsilon(X') \circ G(f') \leftarrow f'.$$

Note that the data of such a functorial isomorphism provides a structure of adjoint pair.

4.1.2. Let  $(H, H^\vee)$  be an adjoint pair of functors, with  $H : \mathcal{C} \rightarrow \mathcal{C}'$ . Let  $\phi \in \text{Hom}(G, H)$ . Then, we define  $\phi^\vee : H^\vee \rightarrow G^\vee$  as the composition

$$\phi^\vee : H^\vee \xrightarrow{\eta_G \mathbf{1}_{H^\vee}} G^\vee G H^\vee \xrightarrow{\mathbf{1}_{G^\vee} \phi \mathbf{1}_{H^\vee}} G^\vee H H^\vee \xrightarrow{\mathbf{1}_{G^\vee} \varepsilon_H} G^\vee.$$

This is the unique map making the following diagram commutative, for any  $X \in \mathcal{C}$  and  $X' \in \mathcal{C}'$  :

$$\begin{array}{ccc} \mathrm{Hom}(HX, X') & \xrightarrow{\mathrm{Hom}(\phi(X), X')} & \mathrm{Hom}(GX, X') \\ \gamma_H(X, X') \downarrow \sim & & \sim \downarrow \gamma_G(X, X') \\ \mathrm{Hom}(X, H^\vee X') & \xrightarrow{\mathrm{Hom}(X, \phi^\vee(X'))} & \mathrm{Hom}(X, G^\vee X') \end{array}$$

We have an isomorphism  $\mathrm{Hom}(G, H) \xrightarrow{\sim} \mathrm{Hom}(H^\vee, G^\vee), \phi \mapsto \phi^\vee$ . We obtain in particular an isomorphism of monoids  $\mathrm{End}(G) \xrightarrow{\sim} \mathrm{End}(G^\vee)^{\mathrm{opp}}$ . Given  $f \in \mathrm{End}(G)$ , then the following diagrams commute

$$\begin{array}{ccc} & G^\vee G & \\ \eta \nearrow & & \searrow 1_{G^\vee} f \\ \mathrm{Id}_{\mathcal{C}} & & G^\vee G \\ \eta \searrow & & \nearrow f^\vee 1_G \\ & G^\vee G & \end{array} \quad \begin{array}{ccc} & GG^\vee & \\ f 1_{G^\vee} \nearrow & & \searrow \varepsilon \\ GG^\vee & & \mathrm{Id}_{\mathcal{C}'} \\ 1_G f^\vee \searrow & & \nearrow \varepsilon \\ & GG^\vee & \end{array}$$

4.1.3. Let now  $(G_1, G_1^\vee)$  and  $(G_2, G_2^\vee)$  be two pairs of adjoint functors, with  $G_1 : \mathcal{C}' \rightarrow \mathcal{C}''$  and  $G_2 : \mathcal{C} \rightarrow \mathcal{C}'$ . The composite morphisms

$$\mathrm{Id}_{\mathcal{C}} \xrightarrow{\eta_2} G_2^\vee G_2 \xrightarrow{1_{G_2^\vee} \eta_1 1_{G_2}} G_2^\vee G_1^\vee G_1 G_2 \quad \text{and} \quad G_1 G_2 G_2^\vee G_1^\vee \xrightarrow{1_{G_1} \varepsilon_2 1_{G_1^\vee}} G_1 G_1^\vee \xrightarrow{\varepsilon_1} \mathrm{Id}_{\mathcal{C}}$$

give an adjoint pair  $(G_1 G_2, G_2^\vee G_1^\vee)$ .

4.1.4. Let  $F = 0 \rightarrow F^r \xrightarrow{d^r} F^{r+1} \rightarrow \dots \rightarrow F^s \rightarrow 0$  be a complex of functors from  $\mathcal{C}$  to  $\mathcal{C}'$  (with  $F^i$  in degree  $i$ ). This defines a functor  $\mathrm{Comp}(\mathcal{C}) \rightarrow \mathrm{Comp}(\mathcal{C}')$  by taking total complexes.

Let  $(F^i, F^{i\vee})$  be adjoint pairs for  $r \leq i \leq s$ . Let

$$F^\vee = 0 \rightarrow F^{s\vee} \xrightarrow{(d^{s-1})^\vee} \dots \rightarrow F^{r\vee} \rightarrow 0$$

with  $F^{i\vee}$  in degree  $-i$ . This complex of functors defines a functor  $\mathrm{Comp}(\mathcal{C}') \rightarrow \mathrm{Comp}(\mathcal{C})$ .

There is an adjunction  $(F, F^\vee)$  between functors on categories of complexes, uniquely determined by the property that given  $X \in \mathcal{C}$  and  $X' \in \mathcal{C}'$ , then  $\gamma_F(X, X') : \mathrm{Hom}_{\mathrm{Comp}(\mathcal{C}')} (FX, X') \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Comp}(\mathcal{C})} (X, F^\vee X')$  is the restriction of

$$\sum_i \gamma_{F^i}(X, X') : \bigoplus_i \mathrm{Hom}_{\mathcal{C}'} (F^i X, X') \xrightarrow{\sim} \bigoplus_i \mathrm{Hom}_{\mathcal{C}} (X, F^{i\vee} X').$$

This extends to the case where  $F$  is unbounded, under the assumption that for any  $X \in \mathcal{C}$ , then  $F^r(X) = 0$  for  $|r| \gg 0$  and for any  $X' \in \mathcal{C}'$ , then  $F^{r\vee}(X') = 0$  for  $|r| \gg 0$ .

4.1.5. Assume  $\mathcal{C}$  and  $\mathcal{C}'$  are abelian categories.

Let  $c \in \mathrm{End}(G)$ . We put  $cG = \mathrm{im}(c)$ . We assume the canonical surjection  $G \rightarrow cG$  splits (i.e.,  $cG = eG$  for some idempotent  $e \in \mathrm{End}(G)$ ). Then, the canonical injection  $c^\vee G^\vee \rightarrow G^\vee$  splits as well (indeed,  $c^\vee G^\vee = e^\vee G^\vee$ ).

Let  $X \in \mathcal{C}$ ,  $X' \in \mathcal{C}'$  and  $\phi \in \text{Hom}(cGX, X')$ . There is  $\psi \in \text{Hom}(GX, X')$  such that  $\phi = \psi|_{cGX}$ . We have a commutative diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{\eta} & G^\vee GX & \xrightarrow{G^\vee c} & G^\vee GX & \xrightarrow{G^\vee \psi} & G^\vee X' \\
 & \searrow \eta & & \nearrow c^\vee G & & \nearrow c^\vee & \\
 & & G^\vee GX & \xrightarrow{G^\vee \psi} & G^\vee X' & & 
 \end{array}$$

It follows that there is a (unique) map  $\gamma_{cG}(X, X') : \text{Hom}(cGX, X') \rightarrow \text{Hom}(X, c^\vee G^\vee X')$  making the following diagram commutative

$$\begin{array}{ccc}
 \text{Hom}(GX, X') & \xrightarrow[\sim]{\gamma_G(X, X')} & \text{Hom}(X, G^\vee X') \\
 \uparrow & & \uparrow \\
 \text{Hom}(cGX, X') & \xrightarrow[\gamma_{cG}(X, X')]{\sim} & \text{Hom}(X, c^\vee G^\vee X')
 \end{array}$$

where the vertical maps come from the canonical projection  $G \rightarrow cG$  and injection  $c^\vee G^\vee \rightarrow G^\vee$ .

Similarly, one shows there is a (unique) map  $\gamma'_{cG}(X, X') : \text{Hom}(X, c^\vee G^\vee X') \rightarrow \text{Hom}(cGX, X')$  making the following diagram commutative

$$\begin{array}{ccc}
 \text{Hom}(GX, X') & \xleftarrow[\sim]{\gamma_G(X, X')^{-1}} & \text{Hom}(X, G^\vee X') \\
 \uparrow & & \uparrow \\
 \text{Hom}(cGX, X') & \xleftarrow[\gamma'_{cG}(X, X')]{\sim} & \text{Hom}(X, c^\vee G^\vee X')
 \end{array}$$

The maps  $\gamma_{cG}(X, X')$  and  $\gamma'_{cG}(X, X')$  are inverse to each other and they provide  $(cG, c^\vee G^\vee)$  with the structure of an adjoint pair. If  $p : G \rightarrow cG$  denotes the canonical surjection, then  $p^\vee : c^\vee G^\vee \rightarrow G^\vee$  is the canonical injection.

4.1.6. Let  $\mathcal{C}$ ,  $\mathcal{C}'$ ,  $\mathcal{D}$  and  $\mathcal{D}'$  be four categories,  $G : \mathcal{C} \rightarrow \mathcal{C}'$ ,  $G^\vee : \mathcal{C}' \rightarrow \mathcal{C}$ ,  $H : \mathcal{D} \rightarrow \mathcal{D}'$  and  $H^\vee : \mathcal{D}' \rightarrow \mathcal{D}$ , and  $(G, G^\vee)$  and  $(H, H^\vee)$  be two adjoint pairs. Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $F' : \mathcal{C}' \rightarrow \mathcal{D}'$  be two fully faithful functors and  $\phi : F'G \xrightarrow{\sim} HF$  be an isomorphism.

We have isomorphisms

$$\begin{aligned}
 \text{Hom}(GG^\vee, \text{Id}_{\mathcal{C}'}) &\xrightarrow[\sim]{F'} \text{Hom}(F'GG^\vee, F') \xrightarrow[\sim]{\text{Hom}(\phi^{-1}\mathbf{1}_{G^\vee}, F')} \text{Hom}(HFG^\vee, F') \\
 &\xrightarrow[\sim]{\gamma_H(FG^\vee, F')} \text{Hom}(FG^\vee, H^\vee F')
 \end{aligned}$$

and let  $\psi : FG^\vee \rightarrow H^\vee F'$  denote the image of  $\varepsilon_G$  under this sequence of isomorphisms.

Then,  $\psi$  is an isomorphism and we have a commutative diagram

$$\begin{array}{ccc}
 F'GG^\vee & \xrightarrow{\mathbf{1}_F \varepsilon_G} & F' \\
 \phi \mathbf{1}_{G^\vee} \downarrow & & \uparrow \varepsilon_H \mathbf{1}_{F'} \\
 HFG^\vee & \xrightarrow{\mathbf{1}_H \psi} & HH^\vee F'
 \end{array}$$

4.2. **Representations of  $\mathfrak{sl}_2$ .** We put

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad h = ef - fe = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp(-f) \exp(e) \exp(-f)$$

$$s^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp(f) \exp(-e) \exp(f)$$

We put  $e_+ = e$  and  $e_- = f$ .

Let  $V$  be a locally finite representation of  $\mathfrak{sl}_2(\mathbf{Q})$  (i.e., a direct sum of finite dimensional representations). Given  $\lambda \in \mathbf{Z}$ , we denote by  $V_\lambda$  the weight space of  $V$  for the weight  $\lambda$  (i.e., the  $\lambda$ -eigenspace of  $h$ ).

For  $v \in V$ , let  $h_\pm(v) = \max\{i | e_\pm^i v \neq 0\}$  and  $d(v) = h_+(v) + h_-(v) + 1$ .

**Lemma 4.1.** *Assume  $V$  is a direct sum of isomorphic simple  $\mathfrak{sl}_2(\mathbf{Q})$ -modules of dimension  $d$ .*

*Let  $v \in V_\lambda$ . Then,*

- $d(v) = d = 1 + 2h_\pm \pm \lambda$
- $e_\mp^{(j)} e_\pm^{(j)} v = \binom{h_\mp + j}{j} \cdot \binom{h_\pm}{j} v$  for  $0 \leq j \leq h_\pm$ .

**Lemma 4.2.** *Let  $\lambda \in \mathbf{Z}$  and  $v \in V_{-\lambda}$ . Then,*

$$s(v) = \sum_{r=\max(0, -\lambda)}^{h_-(v)} \frac{(-1)^r}{r!(\lambda + r)!} e^{\lambda+r} f^r(v) \quad \text{and} \quad s^{-1}(v) = \sum_{r=\max(0, \lambda)}^{h_+(v)} \frac{(-1)^r}{r!(-\lambda + r)!} e^r f^{-\lambda+r}(v).$$

In the following Lemma, we investigate bases of weight vectors with positivity properties.

**Lemma 4.3.** *Let  $V$  be a locally finite  $\mathfrak{sl}_2(\mathbf{Q})$ -module. Let  $\mathcal{B}$  be a basis of  $V$  consisting of weight vectors such that  $\bigoplus_{b \in \mathcal{B}} \mathbf{Q}_{\geq 0} b$  is stable under the actions of  $e_+$  and  $e_-$ . Let  $\mathcal{L}_\pm = \{b \in \mathcal{B} | e_\mp b = 0\}$  and given  $r \geq 0$ , let  $V^{\leq r} = \bigoplus_{d(b) \leq r} \mathbf{Q} b$ .*

*Then,*

- (1) *Given  $r \geq 0$ , then  $V^{\leq r}$  is a submodule of  $V$  isomorphic to a sum of modules of dimension  $\leq r$ .*
- (2) *Given  $b \in \mathcal{B}$ , we have  $e_\pm^{h_\pm(b)} b \in \mathbf{Q}_{\geq 0} \mathcal{L}_\mp$ .*
- (3) *Given  $b \in \mathcal{L}_\pm$ , there is  $\alpha_b \in \mathbf{Q}_{>0}$  such that  $\alpha_b^{-1} e_\pm^{h_\pm(b)} b \in \mathcal{L}_\mp$  and the map  $b \mapsto \alpha_b^{-1} e_\pm^{h_\pm(b)} b$  is a bijection  $\mathcal{L}_\pm \xrightarrow{\sim} \mathcal{L}_\mp$ .*

*The following assertions are equivalent:*

- (i) *Given  $r \geq 0$ , then  $V^{\leq r}$  is the sum of all the simple submodules of  $V$  of dimension  $\leq r$ .*
- (ii)  *$\{e_\pm^i b\}_{b \in \mathcal{L}_\pm, 0 \leq i \leq h_\pm(b)}$  is a basis of  $V$ .*
- (iii)  *$\{e_\pm^i b\}_{b \in \mathcal{L}_\pm, 0 \leq i \leq h_\pm(b)}$  generates  $V$ .*

*Proof.* Let  $b \in \mathcal{B}$ . We have  $eb = \sum_{c \in \mathcal{B}} u_c c$  with  $u_c \geq 0$ . We have  $0 = e^{h_+(b)} eb = \sum_c u_c e^{h_+(b)} c$  and  $e^{h_+(b)} c \in \bigoplus_{b' \in \mathcal{B}} \mathbf{Q}_{\geq 0} b'$ , hence  $e^{h_+(b)} c = 0$  for all  $c \in \mathcal{B}$  such that  $u_c \neq 0$ . So,  $h_+(c) \leq h_+(b)$  for all  $c \in \mathcal{B}$  such that  $u_c \neq 0$ . Hence, (1) holds.

We have  $e_\pm^{h_\pm(b)} b = \sum_{c \in \mathcal{B}} v_c c$  with  $v_c \geq 0$ . Since  $\sum_{c \in \mathcal{B}} v_c e_\pm c = 0$  and  $e_\pm c \in \bigoplus_{b' \in \mathcal{B}} \mathbf{Q}_{\geq 0} b'$ , it follows that  $e_\pm c = 0$  for all  $c$  such that  $v_c \neq 0$ , hence (2) holds.

Let  $b \in \mathcal{L}_\pm$ . We have  $e_\pm^{h_\pm(b)}b = \sum_{c \in \mathcal{B}} v_c c$  with  $v_c \geq 0$ . We have  $e_\mp^{h_\pm(b)}e_\pm^{h_\pm(b)}b = \beta b$  for some  $\beta > 0$ . So,  $\sum_{c \in \mathcal{B}} v_c e_\mp^{h_\pm(b)}c = \beta b$ . It follows that given  $c \in \mathcal{B}$  with  $v_c \neq 0$ , there is  $\beta_c \geq 0$  with  $e_\mp^{h_\pm(b)}c = \beta_c b$ . Since  $h_\pm(c) = h_\mp(b)$ , then  $e_\pm^{h_\pm(b)}e_\mp^{h_\pm(b)}c = \beta_c e_\pm^{h_\pm(b)}b$  is a non-zero multiple of  $c$ , and it follows that there is a unique  $c$  such that  $v_c \neq 0$ . This shows (3).

Assume (i). We prove by induction on  $r$  that  $\{e_\pm^i b\}_{b \in \mathcal{L}_\pm, 0 \leq i \leq h_\pm(b) < r}$  is a basis of  $V^{\leq r}$  (this is obvious for  $r = 0$ ). Assume it holds for  $r = d$ . The image of  $\{b \in \mathcal{B} | d(b) = d + 1\}$  in  $V^{\leq d+1}/V^{\leq d}$  is a basis. This module is a multiple of the simple module of dimension  $d + 1$  and  $\{b \in \mathcal{L}_\pm | d(b) = d + 1\}$  maps to a basis of the lowest (resp. highest) weight space of  $V^{\leq d+1}/V^{\leq d}$  if  $\pm = +$  (resp.  $\pm = -$ ). It follows that  $\{e_\pm^i b\}_{b \in \mathcal{L}_\pm, 0 \leq i \leq d = h_\pm(b)}$  maps to a basis of  $V^{\leq d+1}/V^{\leq d}$ . By induction, it follows that  $\{e_\pm^i b\}_{b \in \mathcal{L}_\pm, 0 \leq i \leq h_\pm(b) \leq d}$  is a basis of  $V^{\leq d+1}$ . This proves (ii).

Assume (ii). Let  $v$  be a weight vector with weight  $\lambda$ . We have  $v = \sum_{b \in \mathcal{L}_\pm, 2i = \lambda \pm h_\pm(b)} u_{b,i} e_\pm^i b$  for some  $u_{b,i} \in \mathbf{Q}$ . Take  $s$  maximal such that there is  $b \in \mathcal{L}_\pm$  with  $h_\pm(b) = s + i$  and  $u_{b,i} \neq 0$ . Then,  $e_\pm^s v = \sum_{b \in \mathcal{L}_\pm, i = h_\pm(b) - s} u_{b,i} e_\pm^{h_\pm(b)} b$ . Since the  $e_\pm^{h_\pm(b)} b$  for  $b \in \mathcal{L}_\pm$  are linearly independent, it follows that  $e_\pm^s v \neq 0$ , hence  $s \leq h_+(v)$ . So, if  $d(v) < r$ , then  $h_\pm(b) < r$  for all  $b$  such that  $u_{b,i} \neq 0$ . We deduce that (i) holds.

The equivalence of (ii) and (iii) is an elementary fact of representation theory of  $\mathfrak{sl}_2(\mathbf{Q})$ .  $\square$

## 5. $\mathfrak{sl}_2$ -CATEGORIFICATION

### 5.1. Weak categorifications.

5.1.1. Let  $\mathcal{A}$  be an artinian and noetherian  $k$ -linear abelian category with the property that the endomorphism ring of any simple object is  $k$  (i.e., every object of  $\mathcal{A}$  is a successive extension of finitely many simple objects and the endomorphism ring of a simple object is  $k$ ).

A *weak  $\mathfrak{sl}_2$ -categorification* is the data of an adjoint pair  $(E, F)$  of exact endo-functors of  $\mathcal{A}$  such that

- the action of  $e = [E]$  and  $f = [F]$  on  $V = \mathbf{Q} \otimes K_0(\mathcal{A})$  gives a locally finite  $\mathfrak{sl}_2$ -representation
- the classes of the simple objects of  $\mathcal{A}$  are weight vectors
- $F$  is isomorphic to a left adjoint of  $E$ .

We denote by  $\varepsilon : EF \rightarrow \text{Id}$  and  $\eta : \text{Id} \rightarrow FE$  the (fixed) counit and unit of the pair  $(E, F)$ . We don't fix an adjunction between  $F$  and  $E$ .

**Remark 5.1.** Assume  $\mathcal{A} = A\text{-mod}$  for a finite dimensional  $k$ -algebra  $A$ . The requirement that  $E$  and  $F$  induce an  $\mathfrak{sl}_2$ -action on  $K_0(\mathcal{A})$  is equivalent to the same condition for  $K_0(\mathcal{A}\text{-proj})$ . Furthermore, the perfect pairing  $K_0(\mathcal{A}\text{-proj}) \times K_0(\mathcal{A}) \rightarrow \mathbf{Z}$ ,  $([P], [S]) \mapsto \dim_k \text{Hom}_{\mathcal{A}}(P, S)$  induces an isomorphism of  $\mathfrak{sl}_2$ -modules between  $K_0(\mathcal{A})$  and the dual of  $K_0(\mathcal{A}\text{-proj})$ .

**Remark 5.2.** A crucial case of application will be  $\mathcal{A} = A\text{-mod}$ , where  $A$  is a symmetric algebra. In that case, the choice of an adjunction  $(E, F)$  determines an adjunction  $(F, E)$ .

We put  $E_+ = E$  and  $E_- = F$ . By the weight space of an object of  $\mathcal{A}$ , we will mean the weight space of its class (whenever this is meaningful).

Note that the opposite category  $\mathcal{A}^{\text{opp}}$  also carries a weak  $\mathfrak{sl}_2$ -categorification.

Fixing an isomorphism between  $F$  and a left adjoint to  $E$  gives another weak categorification, obtained by swapping  $E$  and  $F$ . We call it the *dual* weak categorification.

The *trivial* weak  $\mathfrak{sl}_2$ -categorification on  $\mathcal{A}$  is the one given by  $E = F = 0$ .

5.1.2. Let  $\mathcal{A}$  and  $\mathcal{A}'$  be two weak  $\mathfrak{sl}_2$ -categorifications. A *morphism of weak  $\mathfrak{sl}_2$ -categorifications* from  $\mathcal{A}'$  to  $\mathcal{A}$  is the data of a functor  $R : \mathcal{A}' \rightarrow \mathcal{A}$  and of isomorphisms of functors  $\zeta_{\pm} : RE'_{\pm} \xrightarrow{\sim} E_{\pm}R$  such that the following diagram commutes

$$(4) \quad \begin{array}{ccc} RF' & \xrightarrow{\zeta_-} & FR \\ \eta RF' \downarrow & & \uparrow FR\varepsilon' \\ FERF' & \xrightarrow{F\zeta_+^{-1}F'} & FRE'F' \end{array}$$

Note that  $\zeta_+$  determines  $\zeta_-$ , and conversely (using a commutative diagram equivalent to the one above).

**Lemma 5.3.** *The commutativity of diagram (4) is equivalent to the commutativity of either of the following two diagrams*

$$\begin{array}{ccc} & R & \\ R\eta' \swarrow & & \searrow \eta R \\ RF'E' & \xrightarrow[\zeta_- E']{\sim} FRE' & \xrightarrow[F\zeta_+]{\sim} FER \end{array} \quad \begin{array}{ccc} & R & \\ R\varepsilon' \swarrow & & \searrow \varepsilon R \\ RE'F' & \xrightarrow[\zeta_+ F']{\sim} ERF' & \xrightarrow[E\zeta_-]{\sim} EFR \end{array}$$

*Proof.* Let us assume diagram (4) is commutative. We have a commutative diagram

$$\begin{array}{ccccccc} R & \xrightarrow{\eta R} & FER & \xrightarrow{F\zeta_+^{-1}} & FRE' & & \\ R\eta' \downarrow & & FER\eta' \downarrow & & FRE'\eta' \downarrow & \searrow \text{id} & \\ RF'E' & \xrightarrow{\eta RF'E'} & FERF'E' & \xrightarrow{F\zeta_+^{-1}F'E'} & FRE'F'E' & \xrightarrow{FRE'E'} & FRE' \end{array}$$

$\zeta_- E'$

This shows the commutativity of the first diagram of the Lemma. The proof of commutativity of the second diagram is similar.

Let us now assume the first diagram of the Lemma is commutative. We have a commutative diagram

$$\begin{array}{ccccc} RF' & \xrightarrow{\text{id}} & RF' & \xrightarrow{\zeta_-} & FR \\ & \searrow R\eta'F' & \uparrow RF'\varepsilon' & & \uparrow FR\varepsilon' \\ & & RF'E'F' & & \\ \eta RF' \downarrow & & \searrow \zeta_- E'F' & & \uparrow \\ FERF' & \xrightarrow{F\zeta_+^{-1}F'} & FRE'F' & & \end{array}$$

So, diagram (4) is commutative. The case of the second diagram is similar.  $\square$

Note that  $R$  induces a morphism of  $\mathfrak{sl}_2$ -modules  $K_0(\mathcal{A}'\text{-proj}) \rightarrow K_0(\mathcal{A})$ .

**Remark 5.4.** Let  $\mathcal{A}'$  be a full abelian subcategory of  $\mathcal{A}$  stable under subobjects, quotients, and stable under  $E$  and  $F$ . Then, the canonical functor  $\mathcal{A}' \rightarrow \mathcal{A}$  is a morphism of weak  $\mathfrak{sl}_2$ -categorifications.

5.1.3. We fix now a weak  $\mathfrak{sl}_2$ -categorification on  $\mathcal{A}$  and we investigate the structure of  $\mathcal{A}$ .

**Proposition 5.5.** *Let  $V_\lambda$  be a weight space of  $V$ . Let  $\mathcal{A}_\lambda$  be the full subcategory of  $\mathcal{A}$  of objects whose class is in  $V_\lambda$ . Then,  $\mathcal{A} = \bigoplus_\lambda \mathcal{A}_\lambda$ . So, the class of an indecomposable object of  $\mathcal{A}$  is a weight vector.*

*Proof.* Let  $M$  be an object of  $\mathcal{A}$  with exactly two composition factors  $S_1$  and  $S_2$ . Assume  $S_1$  and  $S_2$  are in different weight spaces. Then, there is  $\varepsilon \in \{\pm\}$  and  $\{i, j\} = \{1, 2\}$  such that  $h_\varepsilon(S_i) > h_\varepsilon(S_j)$ . Let  $r = h_\varepsilon(S_i)$ . We have  $E_\varepsilon^r M \xrightarrow{\sim} E_\varepsilon^r S_i \neq 0$ , hence all the composition factors of  $E_{-\varepsilon}^r E_\varepsilon^r M$  are in the same weight space as  $S_i$ . Now,

$$\text{Hom}(E_{-\varepsilon}^r E_\varepsilon^r M, M) \simeq \text{Hom}(E_\varepsilon^r M, E_\varepsilon^r M) \simeq \text{Hom}(M, E_{-\varepsilon}^r E_\varepsilon^r M)$$

and these spaces are not zero. It follows that  $M$  has a non-zero simple quotient and a non-zero simple submodule in the same weight space as  $S_i$ . So,  $S_i$  is both a submodule and a quotient of  $M$ , hence  $M \simeq S_1 \oplus S_2$ .

We have shown that  $\text{Ext}^1(S, T) = 0$  whenever  $S$  and  $T$  are simple objects in different weight spaces. The proposition follows.  $\square$

Let  $\mathcal{B}$  be the set of classes of simple objects of  $\mathcal{A}$ . This gives a basis of  $V$  and we can apply Lemma 4.3.

We have a categorification of the fact that a locally finite  $\mathfrak{sl}_2$ -module is an increasing union of finite dimensional  $\mathfrak{sl}_2$ -modules:

**Proposition 5.6.** *Let  $M$  be an object of  $\mathcal{A}$ . Then, there is a Serre subcategory  $\mathcal{A}'$  of  $\mathcal{A}$  stable under  $E$  and  $F$ , containing  $M$  and such that  $K_0(\mathcal{A}')$  is finite dimensional.*

*Proof.* Let  $I$  be the set of isomorphism classes of simple objects of  $\mathcal{A}$  that arise as composition factors of  $E^i F^j M$  for some  $i, j$ . Since  $K_0(\mathcal{A})$  is a locally finite  $\mathfrak{sl}_2$ -module, then  $E^i F^j M = 0$  for  $i, j \gg 0$ , hence  $I$  is finite. Now, the Serre subcategory  $\mathcal{A}'$  generated by the objects of  $I$  satisfies the requirement.  $\square$

We have a (weak) generation result for  $D^b(\mathcal{A})$  :

**Lemma 5.7.** *Let  $C \in D^b(\mathcal{A})$  such that  $\text{Hom}_{D^b(\mathcal{A})}(E^i T, C[j]) = 0$  for all  $i \geq 0$ ,  $j \in \mathbf{Z}$  and  $T$  simple object of  $\mathcal{A}$  such that  $FT = 0$ . Then,  $C = 0$ .*

*Proof.* Assume  $C \neq 0$ . Take  $n$  minimal such that  $H^n(C) \neq 0$  and  $S$  simple such that  $\text{Hom}(S, H^n C) \neq 0$ . Let  $i = h_-(S)$  and let  $T$  be a simple submodule of  $F^i S$ . Then,

$$\text{Hom}(E^i T, S) \simeq \text{Hom}(T, F^i S) \neq 0.$$

So,  $\text{Hom}_{D(\mathcal{A})}(E^i T, C[n]) \neq 0$  and we are done, since  $FT = 0$ .  $\square$

There is an obvious analog of Lemma 5.7 using  $\text{Hom}(C[j], F^i T)$  with  $ET = 0$ . Since  $E$  is also a right adjoint of  $F$ , there are similar statements with  $E$  and  $F$  swapped.



**Proposition 5.8.** *Let  $\mathcal{A}'$  be an abelian category and  $G$  be a complex of exact functors from  $\mathcal{A}$  to  $\mathcal{A}'$  that have exact right adjoints. We assume that for any  $M \in \mathcal{A}$  (resp.  $N \in \mathcal{A}'$ ), then  $G^i(M) = 0$  (resp.  $G^{i\vee}(N) = 0$ ) for  $|r| \gg 0$ .*

*Assume  $G(E^i T)$  is acyclic for all  $i \geq 0$  and  $T$  simple object of  $\mathcal{A}$  such that  $FT = 0$ . Then,  $G(C)$  is acyclic for all  $C \in \text{Comp}^b(\mathcal{A})$ .*

*Proof.* Consider the right adjoint complex  $G^\vee$  to  $G$  (cf §4.1.4). We have an isomorphism

$$\text{Hom}_{D^b(\mathcal{A})}(C, G^\vee G(D)) \simeq \text{Hom}_{D^b(\mathcal{A}')} (G(C), G(D))$$

for any  $C, D \in D^b(\mathcal{A})$ . These spaces vanish for  $C = E^i T$  as in the Proposition. By Lemma 5.7, they vanish for all  $C$ . The case  $C = D$  shows that  $G(D)$  is 0 in  $D^b(\mathcal{A}')$ .  $\square$

**Remark 5.9.** Let  $\mathcal{F}$  be the smallest full subcategory of  $\mathcal{A}$  closed under extensions and direct summands and containing  $E^i T$  for all  $i \geq 0$  and  $T$  simple object of  $\mathcal{A}$  such that  $FT = 0$ . Then, in general, not every projective object of  $\mathcal{A}$  is in  $\mathcal{F}$  (cf the case of  $\mathfrak{S}_3$  and  $p = 3$  in §7.1). On the other hand, if the representation  $K_0(\mathcal{A})$  is isotypic, then one shows that every object of  $\mathcal{A}$  is a quotient of an object of  $\mathcal{F}$  and in particular the projective objects of  $\mathcal{A}$  are in  $\mathcal{F}$ .

Let  $V^{\leq d} = \sum_{b \in \mathcal{B}, d(b) \leq d} \mathbf{Q}b$ . Let  $\mathcal{A}^{\leq d}$  be the full Serre subcategory of  $\mathcal{A}$  of objects whose class is in  $V^{\leq d}$ .

Lemma 4.3(1) gives the following Proposition.

**Proposition 5.10.** *The weak  $\mathfrak{sl}_2$ -structure on  $\mathcal{A}$  restricts to one on  $\mathcal{A}^{\leq d}$  and induces one on  $\mathcal{A}/\mathcal{A}^{\leq d}$ .*

So, we have a filtration of  $\mathcal{A}$  as  $0 \subseteq \mathcal{A}^{\leq 1} \subseteq \dots \subseteq \mathcal{A}$  compatible with the weak  $\mathfrak{sl}_2$ -structure. It induces the filtration  $0 \subseteq V^{\leq 1} \subseteq \dots \subseteq V$ . Some aspects of the study of  $\mathcal{A}$  can be reduced to the study of  $\mathcal{A}^{\leq r}/\mathcal{A}^{\leq r-1}$ . This is particularly interesting when  $V^{\leq r}/V^{\leq r-1}$  is a multiple of the  $r$ -dimensional simple module.

5.1.4. We now investigate simple objects and the effect of  $E_\pm$  on them.

**Lemma 5.11.** *Let  $M$  be an object of  $\mathcal{A}$ . Assume that  $d(S) \geq r$  whenever  $S$  is a simple subobject (resp. quotient) of  $M$ . Then,  $d(T) \geq r$  whenever  $T$  is a simple subobject (resp. quotient) of  $E_\pm^i M$ .*

*Proof.* It is enough to consider the case where  $M$  lies in a weight space by Proposition 5.5. Let  $T$  be a simple subobject of  $E_\pm^i M$ . Since  $\text{Hom}(E_\mp^i T, M) \simeq \text{Hom}(T, E_\pm^i M) \neq 0$ , there is  $S$  a simple subobject of  $M$  that is a composition factor of  $E_\mp^i T$ . Hence,  $d(S) \leq d(E_\mp^i T) \leq d(T)$ . The proof for quotients is similar.  $\square$

Let  $\mathcal{C}_r$  be the full subcategory of  $\mathcal{A}^{\leq r}$  with objects  $M$  such that whenever  $S$  is a simple submodule or a simple quotient of  $M$ , then  $d(S) = r$ .

**Lemma 5.12.** *The subcategory  $\mathcal{C}_r$  is stable under  $E_\pm$ .*

*Proof.* It is enough to consider the case where  $M$  lies in a single weight space by Proposition 5.5. Let  $M \in \mathcal{C}_r$  lie in a single weight space. Let  $T$  be a simple submodule of  $E_\pm M$ . By Lemma 5.11, we have  $d(T) \geq r$ . On the other hand,  $d(T) \leq d(E_\pm M) \leq d(M)$ . Hence,  $d(T) = r$ . Similarly, one proves the required property for simple quotients.  $\square$

## 5.2. Categorifications.

5.2.1. An  $\mathfrak{sl}_2$ -categorification is a weak  $\mathfrak{sl}_2$ -categorification with the extra data of  $q \in k^\times$  and  $a \in k$  with  $a \neq 0$  if  $q \neq 1$  and of  $X \in \text{End}(E)$  and  $T \in \text{End}(E^2)$  such that

- $(\mathbf{1}_E T) \circ (T \mathbf{1}_E) \circ (\mathbf{1}_E T) = (T \mathbf{1}_E) \circ (\mathbf{1}_E T) \circ (T \mathbf{1}_E)$
- $(T + \mathbf{1}_{E^2}) \circ (T - q \mathbf{1}_{E^2}) = 0$
- $T \circ (\mathbf{1}_E X) \circ T = \begin{cases} qX \mathbf{1}_E & \text{if } q \neq 1 \\ X \mathbf{1}_E - T & \text{if } q = 1 \end{cases}$
- $X - a$  is locally nilpotent.

Let  $\mathcal{A}$  and  $\mathcal{A}'$  be two  $\mathfrak{sl}_2$ -categorifications. A *morphism of  $\mathfrak{sl}_2$ -categorifications* from  $\mathcal{A}'$  to  $\mathcal{A}$  is a morphism of weak  $\mathfrak{sl}_2$ -categorifications  $(R, \zeta_+, \zeta_-)$  such that  $a' = a$ ,  $q' = q$  and the following diagrams commute

$$(5) \quad \begin{array}{ccc} RE' & \xrightarrow[\sim]{\zeta_+} & ER \\ RX' \downarrow & & \downarrow XR \\ RE' & \xrightarrow[\sim]{\zeta_+} & ER \end{array} \quad \begin{array}{ccc} RE'E' & \xrightarrow[\sim]{\zeta_+ E'} & ERE' \xrightarrow[\sim]{E \zeta_+} EER \\ RT' \downarrow & & \downarrow TR \\ RE'E' & \xrightarrow[\sim]{\zeta_+ E'} & ERE' \xrightarrow[\sim]{E \zeta_+} EER \end{array}$$

5.2.2. We define a morphism  $\gamma_n : H_n \rightarrow \text{End}(E^n)$  by

$$T_i \mapsto \mathbf{1}_{E^{n-i-1}} T \mathbf{1}_{E^{i-1}} \text{ and } X_i \mapsto \mathbf{1}_{E^{n-i}} X \mathbf{1}_{E^{i-1}}.$$

With our assumptions, the  $H_n$ -module  $\text{End}(E^n)$  (given by left multiplication) is in  $\mathcal{N}_n$ .

Let  $\tau \in \{1, \text{sgn}\}$ . We put  $E^{(\tau, n)} = E^n c_n^\tau$ , the image of  $c_n^\tau : E^n \rightarrow E^n$ . Note that the canonical map  $E^n \otimes_{H_n} H_n c_n^\tau \xrightarrow{\sim} E^{(\tau, n)}$  is an isomorphism (cf §3.2.2).

In the context of symmetric groups, the following Lemma is due to Puig. It is an immediate consequence of Proposition 3.5.

**Lemma 5.13.** *The canonical map  $E^{(\tau, n)} \otimes_{P_n^{\text{sgn}}} c_n^\tau H_n \xrightarrow{\sim} E^n$  is an isomorphism. In particular,  $E^n \simeq n! \cdot E^{(\tau, n)}$  and the functor  $E^{(\tau, n)}$  is a direct summand of  $E^n$ .*

We denote by  $E^{(n)}$  one of the two isomorphic functors  $E^{(1, n)}$ ,  $E^{(\text{sgn}, n)}$ .

Using the adjoint pair  $(E, F)$ , we obtain a morphism  $H_n \rightarrow \text{End}(F^n)^{\text{opp}}$  and the definitions and results above have counterparts for  $E$  replaced by  $F$  (cf §4.1.2).

We obtain a structure of  $\mathfrak{sl}_2$ -categorification on the dual as follows. Put  $\tilde{X} = X^{-1}$  when  $q \neq 1$  (resp.  $\tilde{X} = -X$  when  $q = 1$ ). We choose an adjoint pair  $(F, E)$ . Using this adjoint pair, the endomorphisms  $\tilde{X}$  of  $E$  and  $T$  of  $E^2$  provide endomorphisms of  $F$  and  $F^2$ . We take these as the defining endomorphisms for the dual categorification. We define “ $a$ ” for the dual categorification as the inverse (resp. the opposite) of  $a$  for the original categorification.

**Remark 5.14.** The scalar  $a$  can be shifted : given  $\alpha \in k^\times$  when  $q \neq 1$  (resp.  $\alpha \in k$  when  $q = 1$ ), then we can define a new categorification by replacing  $X$  by  $\alpha X$  (resp. by  $X + \alpha \mathbf{1}_E$ ). This changes  $a$  into  $\alpha a$  (resp.  $\alpha + a$ ). So, the scalar  $a$  can always be adjusted to 1 (resp. to 0).

**Remark 5.15.** Assume  $V$  is a multiple of the simple 2-dimensional  $\mathfrak{sl}_2$ -module. Then, a weak  $\mathfrak{sl}_2$ -categorification consists in the data of  $\mathcal{A}_{-1}$  and  $\mathcal{A}_1$  together with inverse equivalences  $E : \mathcal{A}_{-1} \xrightarrow{\sim} \mathcal{A}_1$  and  $F : \mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_{-1}$ . An  $\mathfrak{sl}_2$ -categorification is the additional data of  $q, a$  and  $X \in \text{End}(E) \simeq Z(\mathcal{A}_1)$  with  $X - a$  nilpotent.

**Remark 5.16.** As soon as  $V$  contains a copy of a simple  $\mathfrak{sl}_2$ -module of dimension 3 or more, then  $a$  and  $q$  are determined by  $X$  and  $T$ .

**Example 5.17.** Take for  $V$  the three dimensional irreducible representation of  $\mathfrak{sl}_2$ . Let  $A_{-2} = A_2 = k$  and  $A_0 = k[x]/x^2$ . We put  $\mathcal{A}_i = A_i\text{-mod}$ . On  $\mathcal{A}_{-2}$ , define  $E$  to be induction  $\mathcal{A}_{-2} \rightarrow \mathcal{A}_0$ . On  $\mathcal{A}_0$ ,  $E$  is restriction  $\mathcal{A}_0 \rightarrow \mathcal{A}_2$  and  $F$  is restriction  $\mathcal{A}_0 \rightarrow \mathcal{A}_{-2}$ . On  $\mathcal{A}_2$ , then  $F$  is induction  $\mathcal{A}_2 \rightarrow \mathcal{A}_0$ .

$$k \begin{array}{c} \xrightarrow{\text{Ind}} \\ \xleftarrow{\text{Res}} \end{array} k[x]/x^2 \begin{array}{c} \xrightarrow{\text{Res}} \\ \xleftarrow{\text{Ind}} \end{array} k$$

Let  $q = 1$  and  $a = 0$ . Let  $X$  be the multiplication by  $x$  on  $\text{Res} : \mathcal{A}_0 \rightarrow \mathcal{A}_2$  and multiplication by  $-x$  on  $\text{Ind} : \mathcal{A}_{-2} \rightarrow \mathcal{A}_0$ . Let  $T \in \text{End}_k(k[x]/x^2)$  be the automorphism swapping 1 and  $x$ . This is an  $\mathfrak{sl}_2$ -categorification of the adjoint representation of  $\mathfrak{sl}_2$ . The corresponding weak categorification was constructed in [HueKho].

**Remark 5.18.** Take for  $V$  the three dimensional irreducible representation of  $\mathfrak{sl}_2$ . Let  $A_{-2} = A_2 = k[x]/x^2$  and  $A_0 = k$ . We put  $\mathcal{A}_i = A_i\text{-mod}$ . On  $\mathcal{A}_{-2}$ , then  $E$  is restriction  $\mathcal{A}_{-2} \rightarrow \mathcal{A}_0$ . On  $\mathcal{A}_0$ ,  $E$  is induction  $\mathcal{A}_0 \rightarrow \mathcal{A}_2$  and  $F$  is induction  $\mathcal{A}_0 \rightarrow \mathcal{A}_{-2}$ . On  $\mathcal{A}_2$ , then  $F$  is restriction  $\mathcal{A}_2 \rightarrow \mathcal{A}_0$ .

$$k[x]/x^2 \begin{array}{c} \xrightarrow{\text{Res}} \\ \xleftarrow{\text{Ind}} \end{array} k \begin{array}{c} \xrightarrow{\text{Ind}} \\ \xleftarrow{\text{Res}} \end{array} k[x]/x^2$$

This is a weak  $\mathfrak{sl}_2$ -categorification but not an  $\mathfrak{sl}_2$ -categorification, since  $E^2 : \mathcal{A}_{-2} \rightarrow \mathcal{A}_2$  is  $(k[x]/x^2) \otimes_k -$ , which is an indecomposable functor.

**Remark 5.19.** Let  $A_{-2} = k$ ,  $A_0 = k \times k$  and  $A_2 = k$ . We define  $E$  and  $F$  as the restriction and induction functors in the same way as in Example 5.17. Then,  $V$  is the direct sum of a 3-dimensional simple representation and a 1-dimensional representation. Assume there is  $X \in \text{End}(E)$  and  $T \in \text{End}(E^2)$  giving an  $\mathfrak{sl}_2$ -categorification. We have  $\text{End}(E^2) = \text{End}_k(k^2)$  and  $X1_E = 1_E X = a1_{E^2}$ . But the quotient of  $H_2(q)$  by the relation  $X_1 = X_2 = a$  is zero ! So, we have a contradiction (note here it is crucial to exclude the affine Hecke algebra at  $q = 1$ ). So, this is a weak  $\mathfrak{sl}_2$ -categorification but not an  $\mathfrak{sl}_2$ -categorification (note that we still have  $E^2 \simeq E \oplus E$ ).

**5.3. Minimal categorification.** We introduce here a categorification of the (finite dimensional) simple  $\mathfrak{sl}_2$ -modules.

We fix  $q \in k^\times$  and  $a \in k$  with  $a \neq 0$  if  $q \neq 1$ . Let  $n \geq 0$  and  $B_i = \bar{H}_{i,n}$  for  $0 \leq i \leq n$ .

We put  $\mathcal{A}(n)_\lambda = B_{(\lambda+n)/2}\text{-mod}$  and  $\mathcal{A}(n) = \bigoplus_i B_i\text{-mod}$ . We put  $E = \bigoplus_{i < n} \text{Ind}_{B_i}^{B_{i+1}}$  and  $F = \bigoplus_{i > 0} \text{Res}_{B_{i-1}}^{B_i}$ . Note that the functors  $\text{Ind}_{B_i}^{B_{i+1}} = B_{i+1} \otimes_{B_i} -$  and  $\text{Res}_{B_i}^{B_{i+1}} = B_{i+1} \otimes_{B_{i+1}} -$  are left and right adjoint.

We have  $EF(B_i) \simeq B_i \otimes_{B_{i-1}} B_i \simeq i(n-i+1)B_i$  and  $FE(B_i) \simeq B_{i+1} \simeq (i+1)(n-i)B_i$  as left  $B_i$ -modules (cf §3.3.3). So,  $(ef - fe)([B_i]) = (2i - n)[B_i]$ . Now,  $\mathbf{Q} \otimes K_0(\mathcal{A}(n)_\lambda) = \mathbf{Q}[B_{(\lambda+n)/2}]$ , hence  $ef - fe$  acts on  $K_0(\mathcal{A}(n)_\lambda)$  by  $\lambda$ . It follows that  $e$  and  $f$  induce an action of  $\mathfrak{sl}_2$  on  $K_0(\mathcal{A}(n))$ , hence we have a weak  $\mathfrak{sl}_2$ -categorification.

The image of  $X_{i+1}$  in  $B_{i+1}$  gives an endomorphism of  $\text{Ind}_{B_i}^{B_{i+1}}$  by right multiplication on  $B_{i+1}$ . Taking the sum over all  $i$ , we get an endomorphism  $X$  of  $E$ . Similarly, the image of  $T_{i+1}$  in  $B_{i+2}$  gives an endomorphism of  $\text{Ind}_{B_i}^{B_{i+2}}$  and taking the sum over all  $i$ , we get an endomorphism  $T$  of  $E^2$ .

This provides an  $\mathfrak{sl}_2$ -categorification. The representation on  $K_0(\mathcal{A}(n))$  is the simple  $(n+1)$ -dimensional  $\mathfrak{sl}_2$ -module.

#### 5.4. Link with affine Hecke algebras.

5.4.1. The following Proposition generalizes and strengthens results of Kleshchev [Kl1, Kl2] in the symmetric groups setting and of Grojnowski and Vazirani [GrVa] in the context of cyclotomic Hecke algebras (cf §7.1 and §7.2).

**Proposition 5.20.** *Let  $S$  be a simple object of  $\mathcal{A}$ , let  $n = h_+(S)$  and  $i \leq n$ .*

- (a)  $E^{(n)}S$  is simple.
- (b) *The socle and head of  $E^{(i)}S$  are isomorphic to a simple object  $T$  of  $\mathcal{A}$ . We have isomorphisms of  $(\mathcal{A}, H_i)$ -bimodules:  $\text{soc } E^i S \simeq \text{hd } E^i S \simeq T \otimes K_i$ .*
- (c) *The morphism  $\gamma_i(S) : H_i \rightarrow \text{End}(E^i S)$  factors through  $\bar{H}_{i,n}$  and induces an isomorphism  $\bar{H}_{i,n} \xrightarrow{\sim} \text{End}(E^i S)$ .*

$$\begin{array}{ccc} & H_i & \\ \text{can} \swarrow & & \searrow \gamma_i(S) \\ \bar{H}_{i,n} & \xrightarrow{\sim} & \text{End}(E^i S) \end{array}$$

- (d) *We have  $[E^{(i)}S] - \binom{n}{i}[T] \in V^{\leq d(T)-1}$ .*

*The corresponding statements with  $E$  replaced by  $F$  and  $h_+(S)$  by  $h_-(S)$  hold as well.*

*Proof.* • Let us assume (a) holds. We will show that (b), (c), and (d) follow.

We have  $E^n S \simeq n! \cdot T'$  for some  $T'$  simple. So, we have  $E^n S \simeq T' \otimes R$  as  $(\mathcal{A}, H_n)$ -bimodules, where  $R$  is a right  $H_n$ -module in  $\mathcal{N}_n$ . Since  $\dim R = \dim K_n$ , it follows that  $R \simeq K_n$ .

We have  $E^{n-i} \text{soc } E^{(i)} S \subset E^{n-i} E^{(i)} S \simeq T' \otimes K_n c_i^1$ . Since  $T' \otimes K_n c_i^1$  has a simple socle (Lemma 3.6), it follows that  $E^{n-i} \text{soc } E^{(i)} S$  is an indecomposable  $(\mathcal{A}, H_{n-i})$ -bimodule. If  $T$  is a non-zero summand of  $\text{soc } E^{(i)} S$ , then  $E^{n-i} T \neq 0$  (Lemma 5.12). So,  $T = \text{soc } E^{(i)} S$  is simple. We have  $\text{soc } E^i S \simeq T \otimes R$  for some  $H_i$ -module  $R$  in  $\mathcal{N}_i$ . Since  $\dim R = i!$ , it follows that  $R \simeq K_i$ . The proof for the head is similar.

The dimension of  $\text{End}(E^{(i)} S)$  is at most the multiplicity  $p$  of  $T$  as a composition factor of  $E^{(i)} S$ . Since  $E^{(n-i)} T \neq 0$ , it follows that the dimension of  $\text{End}(E^{(i)} S)$  is at most the number of composition factors of  $E^{(n-i)} E^{(i)} S$ . We have  $E^{(n-i)} E^{(i)} S \simeq \binom{n}{i} \cdot T'$ . So,  $\dim \text{End}(E^{(i)} S) \leq \binom{n}{i}$  and  $\dim \text{End}(E^i S) \leq (i!)^2 \binom{n}{i} = \dim \bar{H}_{i,n}$ .

Since  $\ker \gamma_n(S)$  is a proper ideal of  $H_n$ , we have  $\ker \gamma_n(S) \subset \mathfrak{n}_n H_n$ . We have  $\ker \gamma_i(S) \subset H_i \cap \ker \gamma_n(S) \subset H_i \cap (\mathfrak{n}_n H_n)$ . So, the canonical map  $H_i \rightarrow \bar{H}_{i,n}$  factors through a surjective map:  $\text{im } \gamma_i(S) \twoheadrightarrow \bar{H}_{i,n}$ . We deduce that  $\gamma_i(S)$  is surjective and  $\bar{H}_{i,n} \xrightarrow{\sim} \text{End}(E^i S)$ . So, (c) holds. We deduce also that  $p = \binom{n}{i}$  and that if  $L$  is a composition factor of  $E^{(i)} S$  with  $E^{(n-i)} L \neq 0$ , then  $L \simeq T$ . So, (d) holds. Since the simple object  $\text{hd } E^{(i)} S$  is not killed by  $E^{(n-i)}$  (Lemma 5.12), we deduce that  $\text{hd } E^{(i)} S \simeq T$ . We have now shown (b).

• Let us show that (a) (hence (b), (c), and (d)) holds when  $FS = 0$ . By Lemma 4.3 (3), we have  $[E^{(n)} S] = r[T]$  for some simple object  $T$  and  $r \geq 1$  integer. Since  $[F^{(n)} E^{(n)} S] = [S]$ , we have  $r = 1$ , so (a) holds.

• Let us now show (a) in general. Let  $L$  be a simple quotient of  $F^{(r)} S$ , where  $r = h_-(S)$ . Since  $\text{Hom}(S, E^{(r)} L) \simeq \text{Hom}(F^{(r)} S, L) \neq 0$ , we deduce that  $S$  is isomorphic to a submodule of  $E^{(r)} L$ .

Since  $FL = 0$ , we know by (a) that  $E^{(n)}E^{(r)}L \simeq \binom{n+r}{r}T$  for some simple object  $T$ . So,  $E^{(n)}S \simeq mT$  for some positive integer  $m$ . We have  $\text{Hom}(E^{(n)}S, T) \simeq \text{Hom}(S, F^{(n)}T)$ . Since  $ET = 0$ , we deduce that  $\text{soc } F^{(n)}T$  is simple (we use (b) in its “ $F$ ” version). So,  $\dim \text{Hom}(S, F^{(n)}T) \leq 1$ , hence  $m = 1$  and (a) holds.  $\square$

**Corollary 5.21.** *The  $\mathfrak{sl}_2(\mathbf{Q})$ -module  $V^{\leq d}$  is the sum of the simple submodules of  $V$  of dimension  $\leq d$ .*

*Proof.* Let  $S$  be a simple object of  $\mathcal{A}$  with  $r = h_-(S)$ . By Proposition 5.20 (a),  $T = F^{(r)}S$  is simple. We deduce that  $S \simeq \text{soc } E^{(r)}T$  by adjunction. Now, Proposition 5.20 (d) shows that  $[E^{(r)}T] - \binom{d(S)}{r}[S] \in V^{\leq d(S)-1}$ .

We deduce by induction on  $r$  that  $\{[E^r T]\}$  generates  $V$ , where  $T$  runs over the isomorphism classes of simple objects killed by  $F$  and  $0 \leq r \leq h_+(T)$ . The Corollary follows from Lemma 4.3, (iii)  $\implies$  (i).  $\square$

**Remark 5.22.** Let  $S$  be a simple object of  $\mathcal{A}$  and  $i \leq h_+(S)$ . The action of  $Z_{i,n} = Z(\bar{H}_{i,n})$  on  $E^i S$  restricts to an action on  $E^{(i)}S$ . Since  $E^i S$  is a faithful right  $\bar{H}_{i,n}$ -module, it follows from Proposition 3.5 that  $E^{(i)}S$  is a faithful  $Z_{i,n}$ -module. Now,  $\dim \text{End}_{\mathcal{A}}(E^{(i)}S) = \frac{1}{(i!)^2} \dim \bar{H}_{i,n} = \dim Z_{i,n}$ , hence the morphism  $Z_{i,n} \rightarrow \text{End}_{\mathcal{A}}(E^{(i)}S)$  is an isomorphism.

Let us now continue with the following crucial Lemma whose proof uses some of the ideas of the proof of Proposition 5.20.

**Lemma 5.23.** *Let  $U$  be a simple object of  $\mathcal{A}$  such that  $FU = 0$ . Let  $n = h_+(U)$ ,  $i < n$ , and  $B_i = \bar{H}_{i,n}$ . The composition of  $\eta(E^i U) \otimes 1 : E^i U \otimes_{B_i} B_{i+1} \rightarrow FE^{i+1}U \otimes_{B_i} B_{i+1}$  with the action map  $FE^{i+1}U \otimes_{B_i} B_{i+1} \rightarrow FE^{i+1}U$  is an isomorphism*

$$E^i U \otimes_{B_i} B_{i+1} \xrightarrow{\sim} FE^{i+1}U.$$

*Proof.* By Proposition 3.5, it is enough to prove that the map becomes an isomorphism after applying  $- \otimes_{B_{i+1}} B_{i+1}c_{i+1}^1$ . By (3), we have  $B_{i+1}c_{i+1}^1 = \bigoplus_{a=0}^{n-i-1} \bar{P}_{i,n} x_{i+1}^a c_{i+1}^1$ . Consider the composition

$$\phi = g \circ (f \otimes 1) : E^{(i)}U \otimes \bigoplus_{a=0}^{n-i-1} kx^a \rightarrow FE^{(i+1)}U$$

where  $f : E^{(i)}U \xrightarrow{\eta(E^{(i)}U)} FEE^{(i)}U \xrightarrow{\mathbf{1}_{Fc_{[\mathfrak{S}_i \setminus \mathfrak{S}_{i+1}]}}^1 U} FE^{(i+1)}U$  and  $g : FE^{(i+1)}U \otimes \bigoplus_{a=0}^{n-i-1} kx^a \rightarrow FE^{(i+1)}U$  is given by the action on  $F$ . We have to prove that  $\phi$  is an isomorphism. We have  $[FE^{(i+1)}U] = (n-i)[E^{(i)}U]$ , hence it suffices to prove that  $\phi$  is injective. In order to do that, one may restrict  $\phi$  to a map between the socles of the objects (viewed in  $\mathcal{A}$ ). Let  $\phi_a : \text{soc } E^{(i)}U \rightarrow FE^{(i+1)}U$  be the restriction of  $\phi$  to the socle of  $E^{(i)}U \otimes kx^a$ . Since  $\text{soc}(E^{(i)}U)$  is simple (Proposition 5.20), the problem is to prove that the maps  $\phi_a$  for  $0 \leq a \leq n-i-1$  are linearly independent. By adjunction, it is equivalent to prove that the maps

$$\psi_a : E \text{ soc } E^{(i)}U \xrightarrow{x^a \mathbf{1}_{\text{soc } E^{(i)}U}} E \text{ soc } E^{(i)}U \xrightarrow{c_{[\mathfrak{S}_i \setminus \mathfrak{S}_{i+1}]}^1 U} E^{(i+1)}U$$

are linearly independent.

We have  $\text{soc } E^{i+1}U \simeq S \otimes K_{i+1}$  as  $(\mathcal{A}, H_{i+1})$ -bimodules, where  $S = \text{soc } E^{(i+1)}U$  is simple (Proposition 5.20). Consider the right  $(k[x_{i+1}] \otimes H_i)$ -submodule  $L' = \text{Hom}_{\mathcal{A}}(S, \text{soc}(E \text{ soc } E^i U))$

of  $L = \text{Hom}_{\mathcal{A}}(S, \text{soc } E^{i+1}U)$ . We have  $H_{i+1} = (H_i \otimes P_{[i+1]})H_{i+1}^f$ , hence  $L = L'H_{i+1}^f$  since  $L$  is a simple right  $H_{i+1}$ -module. So,  $L'c_{i+1}^1 = Lc_{i+1}^1$ , hence  $\text{soc}(E \text{ soc } E^i U)c_{i+1}^1 = \text{soc } E^{(i+1)}U$ . In particular, the map  $E \text{ soc } E^{(i)}U \xrightarrow{c_{[\mathfrak{S}_{i+1}/\mathfrak{S}_i]}^1 U} E^{(i+1)}U$  is injective, since  $E \text{ soc } E^{(i)}U$  has a simple socle by Proposition 5.20.

So, we are left with proving that the maps  $E \text{ soc } E^{(i)}U \xrightarrow{X^a \mathbf{1}_{\text{soc } E^{(i)}U}} E \text{ soc } E^{(i)}U$  are linearly independent, *i.e.*, that the restriction of  $\gamma_1(T) : H_1 \rightarrow \text{End}_{\mathcal{A}}(ET)$  to  $\bigoplus_{a=0}^{n-i-1} kX_1^a$  is injective, where  $T = \text{soc } E^{(i)}U$ . Let  $I$  be the kernel of  $\gamma_{n-i}(T) : H_{n-i} \rightarrow \text{End}_{\mathcal{A}}(E^{n-i}T)$ . Then, as in the proof of Proposition 5.20, we have  $I \subset \mathfrak{n}_{n-i}H_{n-i}$ . So,  $\ker \gamma_1 \subset H_1 \cap \mathfrak{n}_{n-i}H_{n-i}$ , hence the canonical map  $\bigoplus_{a=0}^{n-i-1} kX_1^a \rightarrow \text{End}_{\mathcal{A}}(E^{n-i}T)$  is injective (cf (3)) and we are done.  $\square$

5.4.2. We fix  $U$  a simple object of  $\mathcal{A}$  such that  $FU = 0$ . Let  $n = h_+(U)$ . We put  $B_i = \bar{H}_{i,n}$  for  $0 \leq i \leq n$ .

The canonical isomorphisms of functors

$$E(E^i U \otimes_{B_i} -) \xrightarrow{\sim} E^{i+1}U \otimes_{B_i} - \xrightarrow{\sim} E^{i+1}U \otimes_{B_{i+1}} B_{i+1} \otimes_{B_i} -$$

make the following diagram commutative

$$\begin{array}{ccc} B_{i+1}\text{-mod} & \xrightarrow{E^{i+1}U \otimes_{B_{i+1}} -} & \mathcal{A} \\ B_{i+1} \otimes_{B_i} - \uparrow & & \uparrow E \\ B_i\text{-mod} & \xrightarrow{E^i U \otimes_{B_i} -} & \mathcal{A} \end{array}$$

The canonical isomorphism of functors from Lemma 5.23

$$E^i U \otimes_{B_i} B_{i+1} \otimes_{B_{i+1}} - \xrightarrow{\sim} F(E^{i+1}U \otimes_{B_{i+1}} -)$$

make the following diagram commutative

$$\begin{array}{ccc} B_{i+1}\text{-mod} & \xrightarrow{E^{i+1}U \otimes_{B_{i+1}} -} & \mathcal{A} \\ B_{i+1} \otimes_{B_{i+1}} - \downarrow & & \downarrow F \\ B_i\text{-mod} & \xrightarrow{E^i U \otimes_{B_i} -} & \mathcal{A} \end{array}$$

**Theorem 5.24.** *The construction above is a morphism of  $\mathfrak{sl}_2$ -categorifications  $R_U : \mathcal{A}(n) \rightarrow \mathcal{A}$ .*

*Proof.* The commutativity of diagram (4) follows from the very definition of  $\zeta_-$  given by Lemma 5.23. The commutativity of the diagram (5) is obvious.  $\square$

**Remark 5.25.** Let  $I_n$  be the set of isomorphism classes of simple objects  $U$  of  $\mathcal{A}$  such that  $FU = 0$  and  $h_+(U) = n$ . We have a morphism of  $\mathfrak{sl}_2$ -categorifications

$$\sum_{n, U \in I_n} R_U : \bigoplus_{n, U \in I_n} \mathcal{A}(n) \rightarrow \mathcal{A}$$

that is not an equivalence in general but that induces an isomorphism

$$\bigoplus_{n, U \in I_n} \mathbf{Q} \otimes K_0(\mathcal{A}(n)\text{-proj}) \xrightarrow{\sim} \mathbf{Q} \otimes K_0(\mathcal{A})$$

giving a canonical decomposition of  $\mathbf{Q} \otimes K_0(\mathcal{A})$  into simple summands. In that sense, the categorifications  $\mathcal{A}(n)$  are minimal.

The following Proposition is clear.

**Proposition 5.26.** *Assume  $\mathbf{Q} \otimes K_0(\mathcal{A})$  is a simple  $\mathfrak{sl}_2$ -module of dimension  $n + 1$ . Let  $U$  be the unique simple object of  $\mathcal{A}$  with  $FU = 0$ .*

*Then,  $R_U : \mathcal{A}(n) \rightarrow \mathcal{A}$  is an equivalence of categories if and only if  $U$  is projective.*

Note that a categorification corresponding to an isotypic representation needs not be isomorphic to a sum of minimal categorifications (take for example a trivial  $\mathfrak{sl}_2$ -representation).

## 5.5. Decomposition of $[E, F]$ .

5.5.1. Let  $\sigma : EF \rightarrow FE$  be given as the composition

$$EF \xrightarrow{\eta \mathbf{1}_{EF}} FEEF \xrightarrow{\mathbf{1}_F T \mathbf{1}_F} FEEF \xrightarrow{\mathbf{1}_{FE} \varepsilon} FE.$$

The following gives the categorification of the relation  $[e, f] = h$ .

**Theorem 5.27.** *Let  $\lambda \geq 0$ . Then, we have isomorphisms*

$$\sigma + \sum_{j=0}^{\lambda-1} (\mathbf{1}_F X^j) \circ \eta : EF \operatorname{Id}_{\mathcal{A}_{-\lambda}} \oplus \operatorname{Id}_{\mathcal{A}_{-\lambda}}^{\oplus \lambda} \xrightarrow{\sim} FE \operatorname{Id}_{\mathcal{A}_{-\lambda}}$$

and

$$\sigma + \sum_{j=0}^{\lambda-1} \varepsilon \circ (X^j \mathbf{1}_F) : EF \operatorname{Id}_{\mathcal{A}_\lambda} \xrightarrow{\sim} FE \operatorname{Id}_{\mathcal{A}_\lambda} \oplus \operatorname{Id}_{\mathcal{A}_\lambda}^{\oplus \lambda}.$$

*Proof.* By Proposition 5.8, it is enough to check that the maps are isomorphisms after evaluating the functors at  $E^i U$ , where  $i \geq 0$  and  $U$  is a simple object of  $\mathcal{A}_{-\lambda-2i}$  (resp. of  $\mathcal{A}_{\lambda-2i}$ ) such that  $FU = 0$ . Thanks to Lemma 5.3 and Theorem 5.24, we can do this with  $\mathcal{A}$  replaced by a minimal categorification  $\mathcal{A}(n)$  and this is the content of Proposition 5.31 below.  $\square$

In the case of cyclotomic Hecke algebras, Vazirani [Va] had shown that the values of the functors on simple objects are isomorphic.

**Corollary 5.28.** *The functors  $E$  and  $F$  induce an action of  $\mathfrak{sl}_2$  on the Grothendieck group of  $\mathcal{A}$ , viewed as an additive category.*

5.5.2. We put  $\gamma = \begin{cases} (q-1)a & \text{if } q \neq 1 \\ 1 & \text{if } q = 1 \end{cases}$

$$\text{and } m_{ij}(c) = \begin{cases} \sum_{j \leq d_1 < \dots < d_{i-j-c} \leq i-1} T_{d_1} \cdots T_{d_{i-j-c}} & \text{if } c < i-j \\ 1 & \text{if } c = i-j \\ 0 & \text{if } c > i-j. \end{cases}$$

**Lemma 5.29.** *Let  $j < i$  and  $c \geq 0$ . We have*

$$T_j T_{j+1} \cdots T_{i-1} x_i^c = \gamma^c m_{ij}(c) \pmod{\mathfrak{m}_i H_i}.$$

*In particular,  $T_j T_{j+1} \cdots T_{i-1} x_i^c \in \mathfrak{m}_i H_i$  if  $c > i-j$ .*

*Proof.* By (1), we have

$$T_{i-1}x_i^c - x_{i-1}^c T_{i-1} = \begin{cases} (q-1)(x_i + a)(x_{i-1}^{c-1} + x_{i-1}^{c-2}x_i + \cdots + x_i^{c-1}) & \text{if } q \neq 1 \\ x_{i-1}^{c-1} + x_{i-1}^{c-2}x_i + \cdots + x_i^{c-1} & \text{if } q = 1. \end{cases}$$

Hence

$$T_j T_{j+1} \cdots T_{i-1} x_i^c = T_j T_{j+1} \cdots T_{i-2} x_{i-1}^c T_{i-1} + \gamma T_j T_{j+1} \cdots T_{i-2} x_{i-1}^{c-1} \pmod{\mathfrak{m}_i H_i}.$$

Since  $m_{ij}(c) = m_{i-1,j}(c-1) + m_{i-1,j}(c)T_{i-1}$ , the Lemma follows by induction.  $\square$

**Lemma 5.30.** *Let  $j \leq i$ ,  $c \geq 1$  and  $e = \inf(c-1, i-j)$ . Then, we have*

$$T_j T_{j+1} \cdots T_i x_i^c - T_j T_{j+1} \cdots T_{i-1} x_{i+1}^c T_i = \alpha \left( \gamma^e x_{i+1}^{c-e-1} m_{ij}(e) + \gamma^{e-1} x_{i+1}^{c-e} m_{ij}(e-1) + \cdots + x_{i+1}^{c-1} m_{ij}(0) \right) \pmod{\mathfrak{m}_i H_{i+1}}$$

$$\text{where } \alpha = \begin{cases} (1-q)(x_{i+1} + a) & \text{if } q \neq 1 \\ -1 & \text{if } q = 1. \end{cases}$$

*Proof.* We have

$$T_j T_{j+1} \cdots T_i x_i^c - T_j T_{j+1} \cdots T_{i-1} x_{i+1}^c T_i = \alpha T_j \cdots T_{i-1} (x_i^{c-1} + \cdots + x_{i+1}^{c-1})$$

and the result follows from Lemma 5.29.  $\square$

The following is a Mackey decomposition for the algebras  $B_i = \bar{H}_{i,n}$ .

**Proposition 5.31.** *Let  $i \leq n/2$ . Then, we have an isomorphism of  $(B_i, B_i)$ -bimodules*

$$B_i \otimes_{B_{i-1}} B_i \oplus B_i^{\oplus n-2i} \xrightarrow{\sim} B_{i+1} \\ (b \otimes b', b_1, \dots, b_{n-2i}) \mapsto b T_i b' + \sum_{j=1}^{n-2i} b_j X_{i+1}^{j-1}.$$

Let now  $i \geq n/2$ . Then, we have an isomorphism of  $(B_i, B_i)$ -bimodules

$$B_i \otimes_{B_{i-1}} B_i \xrightarrow{\sim} B_{i+1} \oplus B_i^{\oplus 2i-n} \\ b \otimes b' \mapsto (b T_i b', b b', b X_i b', \dots, b X_i^{2i-n-1} b').$$

*Proof.* Let us consider the first map. We know already that both sides are free  $B_i$ -modules of the same rank (cf §5.3), hence it is enough to show surjectivity.

Let  $M = (P_i/\mathfrak{m}_i) \otimes_{P_i} B_{i+1}$ . This is a right  $B_i$ -module quotient of  $B_{i+1}$ . Let  $L$  be the right  $B_i$ -submodule of  $M$  generated by  $B_i T_i + \sum_{j=0}^{n-2i-1} X_{i+1}^j k$ . The first isomorphism will follow from the proof that  $M = L$ . From now on, all elements are viewed in  $M$ .

We have

$$x_{i+1}^{n-i} = \sum_{j=0}^{n-i-1} (-1)^{n-i-1+j} x_{i+1}^j e_{n-i-j}(x_{i+1}, \dots, x_n).$$

Given  $r \geq 2$  and  $j \leq n-i-1$ , we have

$$e_{n-i-j}(x_r, \dots, x_n) = e_{n-i-j}(x_{r-1}, x_r, \dots, x_n) - x_{r-1} e_{n-i-j-1}(x_r, \dots, x_n).$$

Since  $e_{n-i-j}(x_1, \dots, x_n) = 0$ , it follows that  $e_{n-i-j}(x_{i+1}, \dots, x_n) = 0$ . So, we have  $x_{i+1}^{n-i} = 0$ .



Take  $1 \leq r \leq i$ . Then,  $r \leq n - i$  and we have (Lemma 5.30)

$$T_{i-r+1}T_{i-r+2} \cdots T_i x_i^{n-i} = x_{i+1}^{n-i} T_{i-r+1} \cdots T_i + \alpha \left( \gamma^{r-1} x_{i+1}^{n-i-r} + \gamma^{r-2} x_{i+1}^{n-i-r+1} m_{i,i-r+1}(r-2) + \cdots + x_{i+1}^{n-i-1} m_{i,i-r+1}(0) \right).$$

So,

$$T_{i-r+1}T_{i-r+2} \cdots T_i x_i^{n-i} + \alpha \gamma^{r-1} x_{i+1}^{n-i-r} \in \sum_{j \geq 0} x_{i+1}^{n-i-r+1+j} H_i.$$

Since  $x_{i+1}^{n-i} = 0$ , we deduce by induction on  $r$  that  $x_{i+1}^{n-i-r} \in L$  for  $1 \leq r \leq i$ . So,  $x_{i+1}^a \in L$  for all  $a \geq 0$ . We deduce from Lemma 5.30 that  $x_{i+1}^a T_j \cdots T_i \in L$  for all  $1 \leq j \leq i$  and  $a \geq 0$ . Since  $B_{i+1} = \bigoplus_{0 \leq a \leq n-i-1, w \in [\mathfrak{S}_{i+1}/\mathfrak{S}_i]} \bar{P}_{i,n} x_{i+1}^a T_w H_i^f$  (cf §3.3.1), we obtain finally  $M = L$  and we are done.

Let us now consider the second isomorphism. Let us fix an adjunction  $(F, E)$  with unit  $\eta'$  and counit  $\varepsilon'$  and consider the dual categorification  $\mathcal{A}'$  of  $\mathcal{A}(n)$ . We denote by  $X'$  and  $T'$  its defining endomorphisms. Define  $\sigma' : FE \xrightarrow{\eta' FE} EFFE \xrightarrow{ET'E} EFFE \xrightarrow{EF\varepsilon'} EF$ .

Let  $G = FE$  and  $H = EF$ . There is an adjoint pair  $(EF, EF)$  with counit  $\varepsilon_H : EFEF \xrightarrow{E\varepsilon' F} EF \xrightarrow{\varepsilon} \text{Id}$  and an adjoint pair  $(FE, FE)$  with unit  $\eta_G : \text{Id} \xrightarrow{\eta} FE \xrightarrow{F\eta'E} FEFE$ . Consider the canonical isomorphism

$$\zeta : \text{Hom}(FE, EF) = \text{Hom}(G, H) \xrightarrow{\sim} \text{Hom}(H^\vee, G^\vee) \xrightarrow{\sim} \text{Hom}(EF, FE)$$

corresponding to these adjunctions. The commutativity of the following diagram shows that  $\zeta(\sigma') = \sigma$ .

$$\begin{array}{ccccccc}
& EF & & & & & \\
& \downarrow \eta_{EF} & & & & & \\
FE EF & \xrightarrow{F\eta' EEF} & FE FE EF & \xrightarrow{FE \eta' FEEF} & FE EF FE EF & \xrightarrow{FE ET' EEF} & FE EF FE EF \\
& \searrow F\eta' EEF & & & & & \\
& & FE FE EF & \xrightarrow{FE \eta' FEEF} & FE EF FE EF & \xrightarrow{FT FFE EF} & FE EF FE EF \\
& & \downarrow FE \varepsilon' EF & & \downarrow FEEF \varepsilon' EF & & \downarrow FEEF \varepsilon' EF \\
& & FE EF & \xrightarrow{FE \eta' EF} & FE EF EF & & FE EF EF \\
& & \searrow FEEF & & \downarrow FEE \varepsilon' F & & \downarrow FEE \varepsilon' F \\
& & & & FE EF & \xrightarrow{FTF} & FE EF \\
& & & & & & \downarrow FE \varepsilon \\
& & & & & & FE
\end{array}$$

Similarly, using the canonical adjoint pair  $(\text{Id}, \text{Id})$ , we get a canonical isomorphism

$$\zeta' : \text{Hom}(\text{Id}, EF) = \text{Hom}(\text{Id}, H) \xrightarrow{\sim} \text{Hom}(H^\vee, \text{Id}) \xrightarrow{\sim} \text{Hom}(EF, \text{Id}).$$

We have  $\zeta'((\mathbf{1}_E(\tilde{X}')^j) \circ \eta') = \varepsilon \circ (X^j \mathbf{1}_F)$ .

We have shown that the adjoint to

$$\sigma + \sum_{j=0}^{\lambda-1} \varepsilon \circ (X^j \mathbf{1}_F) : EF \operatorname{Id}_{\mathcal{A}_\lambda} \xrightarrow{\sim} FE \operatorname{Id}_{\mathcal{A}_\lambda} \oplus \operatorname{Id}_{\mathcal{A}_\lambda}^{\oplus \lambda}$$

is

$$\sigma' + \sum_{j=0}^{\lambda-1} (\mathbf{1}_{F'}(\tilde{X}')^j) \circ \eta' : E'F' \operatorname{Id}_{\mathcal{A}'_{-\lambda}} \oplus \operatorname{Id}_{\mathcal{A}'_{-\lambda}}^{\oplus \lambda} \rightarrow F'E' \operatorname{Id}_{\mathcal{A}'_{-\lambda}}.$$

One checks easily that the first map of the Proposition remains an isomorphism if  $X_{i+1}$  is replaced by  $\tilde{X}_{i+1}$ . Since the categorification  $\mathcal{A}'$  is isomorphic to  $\mathcal{A}(n)$ , this shows that the map  $\sigma' + \sum_{j=0}^{\lambda-1} (\mathbf{1}_{F'}(\tilde{X}')^j) \circ \eta'$  is an isomorphism, hence  $\sigma + \sum_{j=0}^{\lambda-1} \varepsilon \circ (X^j \mathbf{1}_F)$  is an isomorphism as well.  $\square$

5.5.3. Let us fix a family  $\{M_\lambda \in \mathcal{A}_\lambda\}_\lambda$ . Let  $\mathcal{M}_\lambda$  be the full subcategory of  $\mathcal{A}_\lambda$  whose objects are finite direct sums of direct summands of  $M_\lambda$ . We assume that  $\mathcal{M} = \bigoplus_\lambda \mathcal{M}_\lambda$  is stable under  $E$  and  $F$ .

Let  $A'_\lambda = \operatorname{End}_{\mathcal{A}}(M_\lambda)$ ,  $\mathcal{A}'_\lambda = A'_\lambda\text{-mod}$  and  $\mathcal{A}' = \bigoplus_\lambda \mathcal{A}'_\lambda$ . We put

$$E' = \bigoplus_\lambda \operatorname{Hom}_{\mathcal{A}}(M_{\lambda+2}, EM_\lambda) \otimes_{A'_\lambda} - : \mathcal{A}' \rightarrow \mathcal{A}'$$

and  $F' = \bigoplus_\lambda \operatorname{Hom}_{\mathcal{A}}(M_{\lambda-2}, FM_\lambda) \otimes_{A'_\lambda} - : \mathcal{A}' \rightarrow \mathcal{A}'.$

We have  $\operatorname{Hom}_{\mathcal{A}}(M_{\lambda+2}, EM_\lambda) \simeq \operatorname{Hom}_{\mathcal{A}}(FM_{\lambda+2}, M_\lambda)$  and  $FM_{\lambda+2} \in \mathcal{M}_\lambda$ . It follows that  $\operatorname{Hom}_{\mathcal{A}}(M_{\lambda+2}, EM_\lambda)$  is a projective right  $A'_\lambda$ -module, so  $E'$  is an exact functor. Similarly,  $F'$  is an exact functor. Also, they send projectives to projectives.

Consider the functor  $R = \bigoplus_\lambda M_\lambda \otimes_{A'_\lambda} - : \mathcal{A}' \rightarrow \mathcal{A}$ . Its restriction to  $\mathcal{A}'\text{-proj}$  is an equivalence  $\mathcal{A}'\text{-proj} \xrightarrow{\sim} \mathcal{M}$ . So, the functor  $G \mapsto RG$  from the category of exact functors  $\mathcal{A}' \rightarrow \mathcal{A}'$  sending projectives to projectives to the category of functors  $\mathcal{A}' \rightarrow \mathcal{A}$  is fully faithful.

The canonical map

$$M_{\lambda+2} \otimes_{A'_{\lambda+2}} \operatorname{Hom}_{\mathcal{A}}(M_{\lambda+2}, EM_\lambda) \xrightarrow{\sim} EM_\lambda, \quad m \otimes f \mapsto f(m)$$

is an isomorphism, since  $EM_\lambda \in \mathcal{M}_{\lambda+2}$ . The induced map

$$M_{\lambda+2} \otimes_{A'_{\lambda+2}} \operatorname{Hom}_{\mathcal{A}}(M_{\lambda+2}, EM_\lambda) \otimes_{A'_\lambda} U \xrightarrow{\sim} E(M_\lambda \otimes_{A'_\lambda} U), \quad m \otimes f \otimes u \mapsto E(m' \mapsto m' \otimes u)(f(m))$$

for  $U \in A'_\lambda\text{-mod}$  is an isomorphism, since it is an isomorphism for  $U = A'_\lambda$ .

We obtain an isomorphism  $RE' \xrightarrow{\sim} ER$  and we construct similarly an isomorphism  $RF' \xrightarrow{\sim} FR$ .

Let  $X'$  (resp.  $T'$ ) be the inverse image of  $X \operatorname{id}_R$  (resp.  $T \operatorname{id}_R$ ) via the canonical isomorphisms  $\operatorname{End}(E') \xrightarrow{\sim} \operatorname{End}(RE') \xrightarrow{\sim} \operatorname{End}(ER)$  (resp.  $\operatorname{End}(E'^2) \xrightarrow{\sim} \operatorname{End}(RE'^2) \xrightarrow{\sim} \operatorname{End}(ERE') \xrightarrow{\sim} \operatorname{End}(E^2R)$ ).

Proceeding similarly, the adjoint pair  $(E, F)$  gives an adjoint pair  $(E', F')$  and the functor  $F'$  is isomorphic to a left adjoint of  $E'$ .

**Theorem 5.32.** *The data above defines an  $\mathfrak{sl}_2$ -categorification on  $\mathcal{A}'$  and a morphism of  $\mathfrak{sl}_2$ -categorifications  $\mathcal{A}' \rightarrow \mathcal{A}$ .*

*Proof.* The  $\mathfrak{sl}_2$ -relations in  $K_0(\mathcal{A}'\text{-proj})$  hold thanks to Theorem 5.27 applied to the restriction of functors to  $\mathcal{M}$ . The local finiteness follows from the case of  $\mathcal{A}$ . The commutativity of the diagrams of Lemma 5.3 follows immediately from the construction of the adjoint pair  $(E', F')$ . This shows that  $\mathcal{A}'$  is a weak categorification and that  $R$  defines a morphism of weak categorifications.

By construction, this weak categorification is a categorification and the morphism of weak categorifications is actually a morphism of categorifications.  $\square$

**Corollary 5.33.** *Let  $M \in \mathcal{A}$ . Then, there exists a finite dimensional algebra  $A$ , an  $\mathfrak{sl}_2$ -categorification on  $A\text{-mod}$  and a morphism of  $\mathfrak{sl}_2$ -categorifications  $R : A\text{-mod} \rightarrow \mathcal{A}$  such that  $M$  is a direct summand of  $R(A)$ .*

*Proof.* Let  $N = \bigoplus_{i,j \geq 0} E^i F^j M$ , a finite sum. Let  $N_\lambda$  be the projection of  $N$  on  $\mathcal{A}_\lambda$ . Now, we can apply the constructions and results above, the stability being provided by Corollary 5.28.  $\square$

## 6. CATEGORIFICATION OF THE REFLECTION

**6.1. Rickard's complexes.** Let  $\lambda \in \mathbf{Z}$ . We construct a complex of functors

$$\Theta_\lambda : \text{Comp}(\mathcal{A}_{-\lambda}) \rightarrow \text{Comp}(\mathcal{A}_\lambda),$$

following Rickard [Ri1] (originally, for blocks of symmetric groups).

We denote by  $(\Theta_\lambda)^{-r}$  the restriction of  $E^{(\text{sgn}, \lambda+r)} F^{(1,r)}$  to  $\mathcal{A}_{-\lambda}$  for  $r, \lambda + r \geq 0$  and we put  $(\Theta_\lambda)^{-r} = 0$  otherwise.

Consider the map

$$f : E^{\lambda+r} F^r = E^{\lambda+r-1} E F F^{r-1} \xrightarrow{1_{E^{\lambda+r-1}} \varepsilon_1 F^{r-1}} E^{\lambda+r-1} F^{r-1}.$$

We have  $E^{(\text{sgn}, \lambda+r)} = E^{\lambda+r} c_{[\mathfrak{S}_{\lambda+r}/\mathfrak{S}_{[2, \lambda+r]}]}^{\text{sgn}} c_{[2, \lambda+r]}^{\text{sgn}} \subseteq E^{(\text{sgn}, \lambda+r-1)} E$  and similarly  $F^{(1,r)} \subseteq F F^{(1, r-1)}$ , hence  $f$  restricts to a map

$$d^{-r} : E^{(\text{sgn}, \lambda+r)} F^{(1,r)} \rightarrow E^{(\text{sgn}, \lambda+r-1)} F^{(1, r-1)}.$$

We put

$$\Theta_\lambda = \cdots \rightarrow (\Theta_\lambda)^{-i} \xrightarrow{d^{-i}} (\Theta_\lambda)^{-i+1} \rightarrow \cdots.$$

**Lemma 6.1.**  *$\Theta_\lambda$  is a complex. The map  $[\Theta_\lambda] : V_{-\lambda} = K_0(\mathcal{A}_{-\lambda}) \rightarrow V_\lambda = K_0(\mathcal{A}_\lambda)$  coincides with the action of  $s$ .*

*Proof.* The map  $d^{1-r} d^{-r}$  is the restriction of  $1_{E^{\lambda+r-2}} \varepsilon_2 1_{F^{r-2}}$ , where  $\varepsilon_2 : E E F F \xrightarrow{1_E \varepsilon_1 F} E F \xrightarrow{\varepsilon} \text{Id}$ . Since  $c_{\lambda+r}^{\text{sgn}} = c_{[\mathfrak{S}_{\lambda+r}/\mathfrak{S}_2]}^{\text{sgn}} c_2^{\text{sgn}}$  and  $c_r^1 = c_2^1 c_{[\mathfrak{S}_2 \setminus \mathfrak{S}_r]}^1$ , it follows that

$$E^{(\text{sgn}, \lambda+r)} F^{(1,r)} \subseteq E^{\lambda+r-2} E^{(\text{sgn}, 2)} F^{(1,2)} F^{r-2}.$$

So, in order to prove that  $d^{1-r} d^{-r} = 0$ , it is enough to show that the composition

$$E^2 F^2 \xrightarrow{c_2^{\text{sgn}} c_2^1} E^2 F^2 \xrightarrow{\varepsilon_2} \text{Id}$$

vanishes, where  $c_2^{\text{sgn}}$  acts on  $E^2$  and  $c_2^1$  acts on  $F^2$ . This composition is equal to the composition  $E^2 F^2 \xrightarrow{(c_2^{\text{sgn}} c_2^1) 1_{F^2}} E^2 F^2 \xrightarrow{\varepsilon_2} \text{Id}$ , where  $c_2^{\text{sgn}} c_2^1$  acts now on  $E^2$ . We are done, since  $c_2^{\text{sgn}} c_2^1 = 0$ .

The last statement is given by Lemma 4.2.  $\square$

**Remark 6.2.** Let  $M \in \mathcal{A}_{-\lambda}$ . Let  $l = \max\{r \geq 0 \mid F^r M \neq 0\}$ , a finite integer. Then,  $(\Theta_\lambda)^{-i}(M) = 0$  when  $i \notin [\max(0, -\lambda), l]$ .

**6.2. Derived equivalence from the simple reflection.** Let  $\Theta = \bigoplus_\lambda \Theta_\lambda$ .

The following Lemma follows easily from Lemma 5.3.

**Lemma 6.3.** *Let  $R : \mathcal{A}' \rightarrow \mathcal{A}$  be a morphism of  $\mathfrak{sl}_2$ -categorifications. Then, there is an isomorphism of complexes of functors  $\Theta R \xrightarrow{\sim} R\Theta'$ .*

We can now state our main Theorem

**Theorem 6.4.** *The complex of functors  $\Theta$  induces a self-equivalence of  $K^b(\mathcal{A})$  and of  $D^b(\mathcal{A})$  and induces by restriction equivalences  $K^b(\mathcal{A}_{-\lambda}) \xrightarrow{\sim} K^b(\mathcal{A}_\lambda)$  and  $D^b(\mathcal{A}_{-\lambda}) \xrightarrow{\sim} D^b(\mathcal{A}_\lambda)$ . Furthermore,  $[\Theta] = s$ .*

**Remark 6.5.** In the context of symmetric groups, the invertibility of  $\Theta_\lambda$  when the complex has only one (resp. two) non-zero term is due to Scopes [Sco] (resp. Rickard [Ri1]).

*Proof of Theorem 6.4.* Since  $E$  and  $F$  have right adjoints, there is a complex of functors  $\Theta_\lambda^\vee$  that gives a right adjoint to  $\Theta_\lambda$  (cf §4.1.4). Let  $\varepsilon : \Theta_\lambda \Theta_\lambda^\vee \rightarrow \text{Id}$  be the counit of adjunction and  $Z$  its cone. So,  $Z$  is a complex of exact functors  $\mathcal{A}_{-\lambda} \rightarrow \mathcal{A}_\lambda$ .

Pick  $U \in \mathcal{A}$  with  $FU = 0$  and  $E^i U \in \mathcal{A}_{-\lambda}$  and put  $n = h_+(U)$ . The fully faithful functor  $R_U : K^b(\mathcal{A}(n)\text{-proj}) \rightarrow K^b(\mathcal{A})$  commutes with  $\Theta_\lambda$  (Lemma 6.3), hence commutes with  $\Theta_\lambda^\vee$  and with  $Z$  (cf §4.1.6). By Theorem 6.6, we have  $Z(E^i U) = 0$ . Now, Proposition 5.8 shows that  $Z(M) = 0$  in  $D^b(\mathcal{A}_{-\lambda})$  for all  $M \in D^b(\mathcal{A}_{-\lambda})$ . So,  $\varepsilon$  is an isomorphism in  $D(\mathcal{A}_{-\lambda})$ . One shows similarly that  $\Theta_\lambda$  has a left inverse in  $D^b(\mathcal{A}_{-\lambda})$ .

Let us now prove that  $\varepsilon$  is still an isomorphism in  $K^b(\mathcal{A}_{-\lambda})$ . Let  $M \in \text{Comp}^b(\mathcal{A}_{-\lambda})$ . By Corollary 5.33, there is a finite dimensional  $k$ -algebra  $A$ , an  $\mathfrak{sl}_2$ -categorification on  $\mathcal{A}' = A\text{-mod}$  and a morphism of  $\mathfrak{sl}_2$ -categorifications  $R : \mathcal{A}' \rightarrow \mathcal{A}$  such that the terms of  $M$  are direct summands of  $R(A)$ . The functor  $R$  induces a fully faithful triangulated functor  $R' : K^b(\mathcal{A}'\text{-proj}) \rightarrow K^b(\mathcal{A})$ . The derived category case of the Theorem shows that  $\varepsilon'$  is an isomorphism in  $K^b(\mathcal{A}'_{-\lambda}\text{-proj}) \xrightarrow{\sim} D^b(\mathcal{A}'_{-\lambda})$ . As above, we deduce that  $\varepsilon$  is an isomorphism in the image of  $R'$ , hence  $\varepsilon(M)$  is an isomorphism in  $K^b(\mathcal{A}_{-\lambda})$ . One proceeds similarly to show that  $\Theta_\lambda$  has a left inverse in  $K^b(\mathcal{A}_{-\lambda})$ .  $\square$

**6.3. Equivalences for the minimal categorification.**

**Theorem 6.6.** *Let  $n \geq 0$  and  $\mathcal{A} = \mathcal{A}(n)$  be the minimal categorification. Fix  $\lambda \geq 0$  and let  $l = \frac{n-\lambda}{2}$ . The homology of the complex of functors  $\Theta_\lambda$  is concentrated in degree  $-l$  and  $H^{-l}\Theta_\lambda : \mathcal{A}_{-\lambda} \xrightarrow{\sim} \mathcal{A}_\lambda$  is an equivalence.*

*Proof.* In order to show that the homology of  $\Theta_\lambda$  is concentrated in degree  $-l$ , it suffices to show that  $\Theta_\lambda(B_l c_l^1)$  is homotopy equivalent to a complex concentrated in degree  $-l$ , since  $B_l c_l^1$  is a progenerator for  $B_l\text{-mod}$ . This is equivalent to the property that  $H^*(C) = 0$  for  $* \neq -l$ , where  $C = c_{n-l}^{\text{sgn}} \bar{H}_{n-l} \otimes_{B_{n-l}} \Theta_\lambda(B_l c_l^1)$ , since  $c_{n-l}^{\text{sgn}} \bar{H}_{n-l}$  is the unique simple right  $B_{n-l}$ -module and  $C^{-r} = 0$  for  $r > l$ .

We have

$$C^{-r} = c_{n-l}^{\text{sgn}} \bar{H}_{n-l} \otimes_{B_{n-l}} B_{n-l} c_{[l-r+1, n-l]}^{\text{sgn}} \otimes_{B_{l-r}} c_{[l-r+1, l]}^1 B_l \otimes_{B_l} B_l c_l^1.$$

Lemma 3.7 gives an isomorphism

$$C^{-r} \xrightarrow{\sim} c_{n-l}^{\text{sgn}} \bar{H}_{n-l} \otimes_{B_{n-l}} B_{n-l} c_{[l-r+1, n-l]}^{\text{sgn}} c_{[1, l-r]}^1 \otimes_k \bigoplus_{\mu \in P(r, n-l)} m_\mu(x_{l-r+1}, \dots, x_l) k.$$

Proposition 3.3 and Lemma 3.4 give isomorphisms

$$\bigoplus_{0 \leq a_1 < \dots < a_{l-r} < n-l} x_1^{a_1} \dots x_{l-r}^{a_{l-r}} k \xrightarrow[\text{can}]{} \Lambda^{\mathfrak{S}_{l-r}}(P_{n-l}^{\mathfrak{S}_{[l-r+1, n-l]}} \otimes_{P_{n-l}^{\mathfrak{S}_{n-l}}} k)$$

$$\begin{aligned} \Lambda^{\mathfrak{S}_{l-r}}(P_{n-l}^{\mathfrak{S}_{[l-r+1, n-l]}} \otimes_{P_{n-l}^{\mathfrak{S}_{n-l}}} k) &\xrightarrow{\sim} c_{n-l}^{\text{sgn}} \bar{H}_{n-l} \otimes_{B_{n-l}} B_{n-l} c_{[l-r+1, n-l]}^{\text{sgn}} c_{[1, l-r]}^1 \\ y &\mapsto c_{n-l}^{\text{sgn}} y \end{aligned}$$

and these induce isomorphisms  $E^{-r} \xrightarrow[\phi^{-r}]{} D^{-r} \xrightarrow[\psi^{-r}]{} C^{-r}$ , where

$$E^{-r} = \bigoplus_{0 \leq a_1 < \dots < a_{l-r} < n-l} x_1^{a_1} \dots x_{l-r}^{a_{l-r}} k \otimes \bigoplus_{\mu \in P(r, n-l)} m_\mu(x_{l-r+1}, \dots, x_l) k$$

$$\text{and } D^{-r} = \Lambda^{\mathfrak{S}_{l-r}}(P_{n-l}^{\mathfrak{S}_{[l-r+1, n-l]}} \otimes_{P_{n-l}^{\mathfrak{S}_{n-l}}} k) \otimes \bigoplus_{\mu \in P(r, n-l)} m_\mu(x_{l-r+1}, \dots, x_l) k$$

Let  $\mu \in P(r, n-l)$  and  $0 \leq a_1 < \dots < a_{l-r} < n-l$ . Given a positive integer  $b$ , we write  $b \prec \mu$  when  $b$  appears in  $\mu$  and we denote then by  $\mu \setminus b$  the partition obtained from  $\mu$  by removing one instance of  $b$ . We have  $m_\mu(x_{l-r+1}, \dots, x_l) = \sum_{b \prec \mu} x_{l-r+1}^b m_{\mu \setminus b}(x_{l-r+2}, \dots, x_l)$ . It follows that

$$d_C^{-r} \psi^{-r} (x_1^{a_1} \dots x_{l-r}^{a_{l-r}} \otimes m_\mu) = \psi^{-r+1} \left( \sum_{b \prec \mu} x_1^{a_1} \dots x_{l-r}^{a_{l-r}} x_{l-r+1}^b \otimes m_{\mu \setminus b} \right).$$

Assume  $b = n-l$ . Since  $x_{l-r+1}^{n-l} \in \mathfrak{n}_{n-l} P_{n-l}$ , it follows that  $x_1^{a_1} \dots x_{l-r}^{a_{l-r}} x_{l-r+1}^b$  is 0 in  $\Lambda^{\mathfrak{S}_{l-r+1}}(P_{n-l}^{\mathfrak{S}_{[l-r+2, n-l]}} \otimes_{P_{n-l}^{\mathfrak{S}_{n-l}}} k)$ . One gets the same conclusion when  $b \in \{a_1, \dots, a_{l-r}\}$ .

So,

$$d_C^{-r} \psi^{-r} \phi^{-r} (x_1^{a_1} \dots x_{l-r}^{a_{l-r}} \otimes m_\mu) = \psi^{-r+1} \phi^{-r+1} \left( \sum_{b \prec \mu, b \notin \{a_1, \dots, a_{l-r}, n-l\}} \text{sgn}(\sigma_b) x_1^{a'_1} \dots x_{l-r+1}^{a'_{l-r+1}} \otimes m_{\mu \setminus b} \right)$$

where  $\sigma_b \in \mathfrak{S}_{l-r+1}$  is the permutation such that, putting  $a_{l-r+1} = b$  and  $a'_j = a_{\sigma_b(j)}$ , we have  $a'_1 < a'_2 < \dots < a'_{l-r+1}$ .

Let  $L = k^{n-l}$ , with canonical basis  $\{e_i\}_{1 \leq i \leq n-l}$ . The Koszul complex  $K$  of  $L$  is a bigraded  $k$ -vector space given by  $K^{p,q} = \Lambda^p L \otimes S^q L$ , with a differential of bidegree  $(-1, 1)$  given by

$$(e_{a_1} \dots e_{a_p}) \otimes x \mapsto \sum_{i=1}^p (-1)^{i+p+1} (e_{a_1} \dots e_{a_{i-1}} e_{a_{i+1}} \dots e_{a_p}) \otimes e_{a_i} x.$$

Its dual  $\text{Hom}_k(K, k)$  is isomorphic to  $J$  defined as follows. We put  $J^{p,q} = \Lambda^p(L^*) \otimes S^q(L^*)$ . Let  $\{f_i\}$  be the dual basis of  $L^*$  and  $f_\mu = f_{\mu(1)} \dots f_{\mu(q)} \in S^q(L^*)$  for  $\mu \in P(q, n-l)$ . Then, the

differential  $d_J : J^{p,q} \rightarrow J^{p+1,q-1}$  is given by

$$(f_{a_1} \cdots f_{a_p}) \otimes f_\mu \mapsto \sum_{b < \mu, a_1 < \cdots < a_i < b < a_{i+1} < \cdots < a_p} (-1)^{i+p} (f_{a_1} \cdots f_{a_i} f_b f_{a_{i+1}} \cdots f_{a_p}) \otimes f_{\mu \setminus b}.$$

The homology of  $J$  is concentrated in bidegree  $(0,0)$  and isomorphic to  $k$ . Note that  $J^{\bullet,q}$  is a graded right  $\Lambda(L^*)$ -module, with action given by right multiplication. This provides  $J$  with the structure of a complex of free graded  $\Lambda(L^*)$ -modules (the degree  $-q$  term is  $J^{\bullet,q}$ ), hence of free graded  $k[f_{n-l}]/(f_{n-l}^2)$ -modules by restriction. So, the  $(-q)$ -th homology group of  $J \otimes_{k[f_{n-l}]/(f_{n-l}^2)} k$  is a one-dimensional graded  $k$ -vector space which is in degree  $q$ . The complexes of vector spaces  $J \otimes_{k[f_{n-l}]/(f_{n-l}^2)} k$  and  $Jf_{n-l}$  are isomorphic, with a shift by one in the grading. The complex  $Jf_{n-l}$  decomposes as the direct sum (over  $i$ ) of the complexes  $\bigoplus_q \Lambda^{i-q}(L^*)f_{n-l} \otimes S^q(L^*)$  and the cohomology of such a complex is concentrated in degree  $-i$ .

We have an isomorphism

$$E^{-r} \xrightarrow{\sim} (\Lambda^{l-r} L^*)f_{n-l} \otimes S^r L^* \subseteq J^{l-r+1,r}, \quad x_1^{a_1} \cdots x_{l-r}^{a_{l-r}} \otimes m_\mu \mapsto (f_{a_1} \cdots f_{a_{l-r}} f_{n-l}) \otimes f_\mu.$$

This induces an isomorphism between  $E$  and the subcomplex  $\bigoplus_r (\Lambda^{l-r} L^*)f_{n-l} \otimes S^r L^*$  of  $J^{l+**+1,-*}$ . It follows that the homology of  $E$  is concentrated in degree  $-l$ .

The complex of functors  $\Theta_{-\lambda}$  is given by tensor product by a bounded complex of  $(B_{n-l}, B_l)$ -bimodules which are projective as  $B_{n-l}$ -modules and as  $B_l$ -modules. The homology of that complex is concentrated in the lowest degree where the complex has a non zero component, hence the homology  $M$  is still projective as a  $B_{n-l}$ -module and as a  $B_l$ -module. Lemma 6.1 shows that  $M \otimes_{B_l} -$  sends the unique simple  $B_l$ -module to the unique simple  $B_{n-l}$ -module. By Morita theory,  $M$  induces an equivalence.  $\square$

## 7. EXAMPLES

In this section §7, the field  $k$  will always be assumed to be big enough so that the simple modules considered are absolutely simple.

In most of our examples,  $\mathfrak{sl}_2$ -categorifications are constructed in families, using the following recipe. We start with left and right adjoint functors  $\hat{E}$  and  $\hat{F}$  on an abelian category  $\mathcal{A}$ , together with  $X \in \text{End}(\hat{E})$  and  $T \in \text{End}(\hat{E}^2)$  satisfying the defining relations of (possibly degenerate) affine Hecke algebras. We obtain for each  $a \in k$  (with  $a \neq 0$  if  $q \neq 1$ ) an  $\mathfrak{sl}_2$ -categorification on  $\mathcal{A}$  given by  $E = E_a$  and  $F = F_a$ , the generalised  $a$ -eigenspaces of  $X$  acting on  $\hat{E}$  and  $\hat{F}$ . While we need to check in each example that  $E$  and  $F$  do indeed give an action of  $\mathfrak{sl}_2$  on  $K_0(\mathcal{A})$ , it is automatic that  $X$  and  $T$  restrict to endomorphisms of  $E$  and  $E^2$  with the desired properties. That  $T$  restricts is a consequence of the identity (a special case of (1))

$$\begin{aligned} & T_1(X_2 - a)^N - (X_1 - a)^N T_1 \\ &= \begin{cases} (q-1)X_2[(X_1 - a)^{N-1} + (X_1 - a)^{N-2}(X_2 - a) + \cdots + (X_2 - a)^{N-1}] & \text{if } q \neq 1 \\ (X_1 - a)^{N-1} + (X_1 - a)^{N-2}(X_2 - a) + \cdots + (X_2 - a)^{N-1} & \text{if } q = 1. \end{cases} \end{aligned}$$

in  $H_2(q)$ .

### 7.1. Symmetric groups.

7.1.1. Let  $p$  be a prime number and  $k = \mathbf{F}_p$ . The quotient of  $H_n(1)$  by the ideal generated by  $X_1$  is the group algebra  $k\mathfrak{S}_n$ . The images of  $T_i$  and  $X_i$  in  $k\mathfrak{S}_n$  are  $s_i = (i, i+1)$  and the Jucys-Murphy element  $L_i = (1, i) + (2, i) + \cdots + (i-1, i)$ .

Let  $a \in k$ . Given  $M$  a  $k\mathfrak{S}_n$ -module, we denote by  $F_{a,n}(M)$  the generalized  $a$ -eigenspace of  $X_n$ . This is a  $k\mathfrak{S}_{n-1}$ -module. We have a decomposition  $\text{Res}_{k\mathfrak{S}_{n-1}}^{k\mathfrak{S}_n} = \bigoplus_{a \in k} F_{a,n}$ . There is a corresponding decomposition  $\text{Ind}_{k\mathfrak{S}_{n-1}}^{k\mathfrak{S}_n} = \bigoplus_{a \in k} E_{a,n}$ , where  $E_{a,n}$  is left and right adjoint to  $F_{a,n}$ . We put  $E_a = \bigoplus_{n \geq 1} E_{a,n}$  and  $F_a = \bigoplus_{n \geq 1} F_{a,n}$ .

Recall the following classical result [LLT].

**Theorem 7.1.** *The functors  $E_a$  and  $F_a$  for  $a \in \mathbf{F}_p$  give rise to an action of the affine Lie algebra  $\widehat{\mathfrak{sl}}_p$  on  $\bigoplus_{n \geq 0} K_0(k\mathfrak{S}_n\text{-mod})$ .*

*The decomposition of  $K_0(k\mathfrak{S}_n\text{-mod})$  in blocks coincides with its decomposition in weight spaces.*

*Two blocks of symmetric groups have the same weight if and only if they are in the same orbit under the adjoint action of the affine Weyl group.*

In particular for each  $a \in \mathbf{F}_p$  the functors  $E_a$  and  $F_a$  give a weak  $\mathfrak{sl}_2$ -categorification on  $\mathcal{A} = \bigoplus_{n \geq 0} k\mathfrak{S}_n\text{-mod}$ .

We denote by  $X$  the endomorphism of  $E_a$  given on  $E_{a,n}$  by right multiplication by  $L_n$  (on the  $(k\mathfrak{S}_n, k\mathfrak{S}_{n-1})$ -bimodule  $k\mathfrak{S}_n$ ). We denote by  $T$  the endomorphism of  $E_a^2$  given on  $E_{a,n}E_{a,n-1}$  by right multiplication by  $s_{n-1}$  (on the  $(k\mathfrak{S}_n, k\mathfrak{S}_{n-2})$ -bimodule  $k\mathfrak{S}_n$ ). This gives an  $\mathfrak{sl}_2$ -categorification on  $\mathcal{A}$  (here,  $q = 1$ ).

7.1.2.

**Theorem 7.2.** *Let  $R = k$  or  $\mathbf{Z}_{(p)}$ . Let  $A$  and  $B$  be two blocks of symmetric groups over  $R$  with isomorphic defect groups. Then,  $A$  and  $B$  are splendidly Rickard equivalent (in particular, they are derived equivalent).*

*Proof.* Two blocks of symmetric groups over  $k$  have isomorphic defect groups if and only if they have equal weights (cf §7.1.3 below). By Theorem 7.1, there is a sequence of blocks  $A_0 = A, A_1, \dots, A_r = B$  such that  $A_j$  is the image of  $A_{j-1}$  by some simple reflection  $\sigma_{a_j}$  of the affine Weyl group. By Theorem 6.4, the complex of functors  $\Theta$  associated with  $a = a_j$  induces a self-equivalence of  $K^b(\mathcal{A})$ . It restricts to a splendid Rickard equivalence between  $A_j$  and  $A_{j+1}$ . By composing these equivalences, we obtain a splendid Rickard equivalence between  $A$  and  $B$  (note that the composition of splendid equivalences can easily be seen to be splendid, cf eg [Rou2, Lemma 2.6]).

The constructions of  $E$  and  $F$  lift uniquely to  $\mathbf{Z}_{(p)}$ :  $\text{Ind}_{\mathbf{Z}_{(p)}\mathfrak{S}_{n-1}}^{\mathbf{Z}_{(p)}\mathfrak{S}_n} = \bigoplus_{a \in k} \tilde{E}_a$ ,  $\text{Res}_{\mathbf{Z}_{(p)}\mathfrak{S}_{n-1}}^{\mathbf{Z}_{(p)}\mathfrak{S}_n} = \bigoplus_{a \in k} \tilde{F}_a$ , where  $\tilde{E}_a \otimes_{\mathbf{Z}_{(p)}} k = E_a$ , where  $\tilde{F}_a \otimes_{\mathbf{Z}_{(p)}} k = F_a$  and  $\tilde{E}_a$  and  $\tilde{F}_a$  are left and right adjoint. We denote by  $\tilde{T}$  the endomorphism of  $\tilde{E}_a^2$  given on  $\tilde{E}_{a,n}\tilde{E}_{a,n-1}$  by the action of  $s_{n-1}$ .

The construction of  $\Theta$  in §6.1 lifts to a complex  $\tilde{\Theta}$  of functors on  $\tilde{\mathcal{A}} = \bigoplus_{n \geq 0} \mathbf{Z}_{(p)}\mathfrak{S}_n\text{-mod}$ . By [Ri3, end of proof of Theorem 5.2], the lift  $\tilde{\Theta}$  of  $\Theta$  is a splendid self Rickard equivalence of  $D^b(\tilde{\mathcal{A}})$  and we conclude as before.  $\square$

**Remark 7.3.** The equivalence depends on the choice of a sequence of simple reflections whose product sends one block to the other. If, as expected, the categorifications of the simple

reflections give rise to a braid group action on the derived category of  $\bigoplus_{n \geq 0} k\mathfrak{S}_n\text{-mod}$ , then one can choose the canonical lifting of the affine Weyl group element in the braid group to get a canonical equivalence.

**Remark 7.4.** Theorem 7.2 gives isomorphisms between Grothendieck groups of the blocks (taken over  $\mathbf{Q}$ ) satisfying certain arithmetical properties (perfect isometries or even isotypies). These arithmetical properties were shown already by Enguehard [En].

**Remark 7.5.** Two blocks of symmetric groups over  $k$  have isomorphic defect groups if and only if they have the same number of simple modules, up to the exception of blocks of weights 0 and 1 for  $p = 2$  — note that a block of weight 0 is simple whereas a block of weight 1 is not simple, so two such blocks are not derived equivalent. So, one can restate Theorem 7.2 as follows :

Let  $A$  and  $B$  be two blocks of symmetric groups over  $k$ . Then,  $A$  and  $B$  are derived equivalent if and only if they have isomorphic defect groups. Assume  $A$  and  $B$  are not simple if  $p = 2$ . Then,  $A$  and  $B$  are derived equivalent if and only if  $\text{rank } K_0(A) = \text{rank } K_0(B)$ .

We can now deduce a proof of Broué's abelian defect group conjecture for blocks of symmetric groups :

**Theorem 7.6.** *Let  $A$  be a block of a symmetric group  $G$  over  $\mathbf{Z}_{(p)}$ ,  $D$  a defect group and  $B$  the corresponding block of  $N_G(D)$ . If  $D$  is abelian, then  $A$  and  $B$  are splendidly Rickard equivalent.*

*Proof.* By [ChKe], there is a block  $A'$  of a symmetric group which is splendidly Morita equivalent to the principal block of  $\mathbf{Z}_{(p)}(\mathfrak{S}_p \wr \mathfrak{S}_w)$ , where  $w$  is the weight of  $A$ . We have a splendid Rickard equivalence between the principal block of  $\mathbf{Z}_{(p)}\mathfrak{S}_p$  and  $\mathbf{Z}_{(p)}N$ , where  $N$  is the normalizer of a Sylow  $p$ -subgroup of  $\mathfrak{S}_p$  by [Rou2, Theorem 1.1]. By [Ma, Theorem 4.3] (cf also [Rou2, Lemma 2.8] for the Rickard/derived equivalence part), we deduce a splendid Rickard equivalence between the principal blocks of  $\mathbf{Z}_{(p)}(\mathfrak{S}_p \wr \mathfrak{S}_w)$  and  $\mathbf{Z}_{(p)}(N \wr \mathfrak{S}_w)$ . Now, we have an isomorphism  $B \simeq \mathbf{Z}_{(p)}(N \wr \mathfrak{S}_w) \otimes B_0$ , where  $B_0$  is a matrix algebra over  $\mathbf{Z}_{(p)}$ , hence there is a splendid Morita equivalence between  $B$  and  $\mathbf{Z}_{(p)}(N \wr \mathfrak{S}_w)$ . So, we obtain a splendid Rickard equivalence between  $B$  and  $A'$ .

By Theorem 7.2, we have a splendid Rickard equivalence between  $A$  and  $A'$  and the Theorem follows.  $\square$

**Remark 7.7.** The existence of an isotypy between  $A$  and  $B$  in Theorem 7.6 was known by [Rou1].

7.1.3. Let us analyze more precisely the categorification.

Given  $\lambda$  a partition of  $m$ , we denote by  $|\lambda| = m$  the size of  $\lambda$ . Let  $\kappa$  be a  $p$ -core and  $n$  an integer such that  $p|(n - |\kappa|)$  and  $n \geq |\kappa|$ . We denote by  $b_{\kappa,n}$  the corresponding block of  $k\mathfrak{S}_n$  (the irreducible characters of that block are associated to the partitions having  $\kappa$  as their  $p$ -core). The integer  $\frac{n-|\kappa|}{p}$  is the weight of the block (this notion of weight is not to be confused with the weights relative to Lie algebra actions).

Let  $\lambda$  be a partition with  $p$ -core  $\kappa$  and  $\lambda'$  a partition obtained from  $\lambda$  by adding an  $a$ -node. Then, the  $p$ -core of  $\lambda'$  depends only on  $\kappa$  and  $a$  and we denote it by  $e_a(\kappa)$ . Similarly, we define  $f_a(\kappa)$  by removing an  $a$ -node.



We will freely identify a functor  $M \otimes -$  with the bimodule  $M$ . We have

$$(6) \quad E_{a,n+1} = \bigoplus_{\kappa} b_{e_a(\kappa),n+1} k\mathfrak{S}_{n+1} b_{\kappa,n}$$

where  $\kappa$  runs over the  $p$ -cores such that  $|\kappa| \leq n$ ,  $|\kappa| \equiv n \pmod{p}$  and  $|e_a(\kappa)| \leq n+1$ .

Let  $b_{\kappa_{-r},l}, b_{\kappa_{-r+2},l+1}, \dots, b_{\kappa_r,l+r}$  be a chain of blocks with  $|f_a(\kappa_{-r})| > l-1$ ,  $|e_a(\kappa_r)| > l+r+1$  and  $f_a(\kappa_j) = \kappa_{j-2}$ .

Put  $n_i = l + (i-r)/2$  and  $B_i = k\mathfrak{S}_{n_i} b_{\kappa_i, n_i}$  for  $-r \leq i \leq r$  and  $i \equiv r \pmod{2}$ .

Let  $\mathcal{A} = \bigoplus_i B_i\text{-mod}$ . The action of  $E = E_a$  and  $F = F_a$  on  $K_0(\mathcal{A})$  gives a representation of  $\mathfrak{sl}_2$ . This gives an  $\mathfrak{sl}_2$ -categorification (here,  $q = 1$ ).

The complex of functors  $\Theta$  restricts to a splendid Rickard equivalence between  $B_i$  and  $B_{-i}$ .

Let us recall some results of the local block theory of symmetric groups (cf [Pu1] or [Br, §2]).

Let  $P$  be a  $p$ -subgroup of  $\mathfrak{S}_n$ . Up to conjugacy, we can assume  $[1, n]^P = [n_P + 1, n]$  for some integer  $n_P$  (we call such a  $P$  a standard  $p$ -subgroup). Then,  $C_{\mathfrak{S}_n}(P) = H \times \mathfrak{S}_{[n_P+1, n]}$  where  $H = C_{\mathfrak{S}_{n_P}}(P)$ . The algebra  $kH$  has a unique block.

Given  $G$  a finite group and  $P$  a  $p$ -subgroup of  $G$ , we denote by  $\text{br}_P : (kG)^P \rightarrow kC_G(P)$  the Brauer morphism (restriction of the morphism of  $k$ -vector spaces  $kG \rightarrow kC_G(P)$  which is the identity on  $C_G(P)$  and 0 on  $G - C_G(P)$ ). We denote by  $\text{Br}_P : kG\text{-mod} \rightarrow kC_G(P)\text{-mod}$  the Brauer functor given by  $M \mapsto M^P / (\sum_{Q < P} \text{Tr}_Q^P M^Q)$ , where  $\text{Tr}_Q^P(x) = \sum_{g \in P/Q} g(x)$ .

We will use the following result of Puig and Marichal

**Theorem 7.8.** *We have*

$$\text{br}_P(b_{\kappa,n}) = \begin{cases} 1 \otimes b_{\kappa, n-n_P} & \text{if } \frac{n-n_P-|\kappa|}{p} \in \mathbf{Z}_{\geq 0} \\ 0 & \text{otherwise.} \end{cases}$$

Note in particular that a standard  $p$ -subgroup  $P$  is a defect group of  $b_{\kappa,n}$  if and only if  $P$  is a Sylow  $p$ -subgroup of  $\mathfrak{S}_{n-|\kappa|}$ . In particular, two blocks of symmetric groups have isomorphic defect groups if and only if they have equal weights.

So, we deduce from (6) and Theorem 7.8 :

**Lemma 7.9.** *We have an isomorphism of  $((kH \otimes k\mathfrak{S}_{n-n_P+i}), (kH \otimes k\mathfrak{S}_{n-n_P-1}))$ -bimodules*

$$\text{Br}_{\Delta P}(E_{a,n+i} \cdots E_{a,n+1} E_{a,n}) \xrightarrow{\sim} kH \otimes E_{a,n-n_P+i} \cdots E_{a,n-n_P+1} E_{a,n-n_P}.$$

For  $i = 1$ , it is compatible with the action of  $T$ .

Let  $P$  be a non-trivial standard  $p$ -subgroup of  $\mathfrak{S}_{n-i}$ . If  $\text{br}_P(b_{\kappa_i, n_i})$  is not 0, then

$$\text{Br}_{\Delta P}(b_{\kappa_{-i}, n_{-i}} \Theta b_{\kappa_i, n_i}) \simeq kH \otimes b_{\kappa_{-i}, n_{-i}-n_P} \Theta b_{\kappa_i, n_i-n_P}.$$

Note that this Lemma permits to deduce a proof of the Rickard equivalence in Theorem 7.2 from that of the derived equivalence, by induction on the size of the defect group : By induction,  $b_{\kappa_{-i}, n_{-i}-n_P} \Theta b_{\kappa_i, n_i-n_P}$  induces a Rickard equivalence. Now,  $\Theta$  induces a derived equivalence, so, it follows from Theorem 7.10 below that  $\Theta$  induces a Rickard equivalence between  $B_i$  and  $B_{-i}$ .

If a splendid complex induces local derived equivalences, then it induces a Rickard equivalence [Rou4, Theorem 5.6] (in a more general version, but whose proof extends with no modification) :

**Theorem 7.10.** *Let  $G$  be a finite group,  $b$  a block of  $kG$  and  $D$  a defect group of  $b$ . We assume  $b$  is of principal type, i.e.,  $\text{br}_D(b)$  is a block of  $kC_G(D)$ . Let  $H$  be a subgroup of  $G$  containing  $D$  and controlling the fusion of  $p$ -subgroups of  $D$ . Let  $c$  be the block of  $kH$  corresponding to  $b$ .*

*Let  $C$  be a bounded complex of  $(kGb, kHc)$ -bimodules. We assume  $C$  is splendid, i.e., the components  $M$  of  $C$  are direct summands of modules  $\text{Ind}_{\Delta D}^{G \times H^\circ} N$ , where  $N$  is a permutation  $\Delta D$ -module.*

*Assume*

- $\text{Br}_{\Delta P}(C)$  induces a Rickard equivalence between  $kC_G(P) \text{br}_P(b)$  and  $kC_H(P) \text{br}_P(c)$  for  $P$  a non trivial  $p$ -subgroup of  $D$  and
- $C$  induces a derived equivalence between  $kGb$  and  $kHc$ .

*Then,  $C$  induces a Rickard equivalence between  $kGb$  and  $kHc$ .*

## 7.2. Cyclotomic Hecke algebras.

7.2.1. We consider here the non-degenerate case  $q \neq 1$ . We fix  $v_1, \dots, v_d \in k^\times$ .

We denote by  $\mathcal{H}_n = \mathcal{H}_n(v, q)$  the quotient of  $H_n(q)$  by the ideal generated by  $(X_1 - v_1) \cdots (X_1 - v_d)$ . This is the Hecke algebra of the complex reflection group  $G(d, 1, n)$  (cf e.g. [Ar2, §13.1]).

The algebra  $\mathcal{H}_n$  is free over  $k$  with basis  $\{X_1^{a_1} \cdots X_n^{a_n} T_w\}_{0 \leq a_i < d, w \in \mathfrak{S}_n}$  [ArKo]. In particular  $\mathcal{H}_{n-1}$  embeds as a subalgebra of  $\mathcal{H}_n$ , and  $\mathcal{H}_n$  is free as a left and as a right  $\mathcal{H}_{n-1}$ -module, for the multiplication action. The algebra  $\mathcal{H}_n$  is symmetric [MalMat].

7.2.2. Let  $a \in k^\times$ . Given  $M$  an  $\mathcal{H}_n$ -module, we denote by  $F_{a,n}M$  the generalized  $a$ -eigenspace of  $X_n$ . This is an  $\mathcal{H}_{n-1}$ -module. We have a decomposition  $\text{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} = \bigoplus_{a \in k^\times} F_{a,n}$ . There is a corresponding decomposition  $\text{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} = \bigoplus_{a \in k^\times} E_{a,n}$ , where  $E_{a,n}$  is left and right adjoint to  $F_{a,n}$ . We put  $E_a = \bigoplus_{n \geq 1} E_{a,n}$  and  $F_a = \bigoplus_{n \geq 1} F_{a,n}$ .

Now fix  $a \in k^\times$ . The functors  $E = E_a$  and  $F = F_a$  give an action of  $\mathfrak{sl}_2$  on  $\bigoplus_{n \geq 0} K_0(\mathcal{H}_n\text{-mod})$  in which the classes of simple modules are weight vectors [Ar2, Theorem 12.5] (only the case where each parameter is a power of  $q$  is considered there, but the proof extends immediately to our more general setting). We obtain an  $\mathfrak{sl}_2$ -categorification on  $\bigoplus_{n \geq 0} \mathcal{H}_n\text{-mod}$ , where the endomorphism  $X$  of  $E$  is given on  $E_{a,n}$  by right multiplication by  $X_n$ , and the endomorphism  $T$  of  $E^2$  is given on  $E_{a,n}E_{a,n-1}$  by right multiplication by  $T_{n-1}$ .

**Remark 7.11.** Let  $e$  be the multiplicative order of  $q$  in  $k^\times$ . Fix  $a_0 \in k^\times$  and let  $I = \{q^m a_0 \mid m \in \mathbb{Z}\}$ . Then the functors  $E_a$  and  $F_a$  for  $a \in I$  define an action of  $\widehat{\mathfrak{sl}}_e$  on  $\bigoplus_{n \geq 0} K_0(\mathcal{H}_n\text{-mod})$ .

7.2.3. Consider here the case  $d = 1$ . Then,  $\mathcal{H}_n = \mathcal{H}_n(1, q)$  is the Hecke algebra of  $\mathfrak{S}_n$ . Let  $e$  be the multiplicative order of  $q$  in  $k$ . We have a notion of weight of a block as in §7.1.1, replacing  $p$  by  $e$  in the definitions.

We obtain a  $q$ -analog of Theorem 7.2:

**Theorem 7.12.** *Assume  $d = 1$ . Let  $A$  be a block of  $\mathcal{H}_n$  and  $B$  a block of  $\mathcal{H}_m$ . Then,  $A$  and  $B$  are derived equivalent if and only if they are Rickard equivalent if and only if they have the same weight.*

**Remark 7.13.** All of the constructions and results of §7.2 hold for degenerate cyclotomic Hecke algebras as well, under the assumption that they are symmetric algebras (which should be provable along the lines of [MalMat]).

### 7.3. General linear groups over a finite field.

7.3.1. Let  $q$  be a prime power,  $n \geq 0$  and  $G_n = \mathrm{GL}_n(q)$ . We assume that  $k$  has characteristic  $\ell > 0$  and  $\ell \nmid q(q-1)$ . Let  $A_n = kG_n b_n$  be the sum of the unipotent blocks of  $kG_n$ .

Given  $H$  a finite group, we put  $e_H = \frac{1}{|H|} \sum_{h \in H} h$ . We denote by  ${}^t g$  the transpose of a matrix  $g$ .

We denote by  $V_n$  the subgroup of upper triangular matrices of  $G_n$  with diagonal coefficients 1 and whose off-diagonal coefficients vanish outside the  $n$ -th column. We denote by  $D_n$  the subgroup of  $G_n$  of diagonal matrices with diagonal entries 1 except the  $(n, n)$ -th one.

$$V_n = \begin{pmatrix} 1 & & * \\ & \ddots & \vdots \\ & & 1 & * \\ & & & 1 \end{pmatrix}, \quad D_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & * \end{pmatrix}$$

Let  $i \leq n$ . We view  $G_i$  as a subgroup of  $G_n$  via the first  $i$  coordinates.

We put

$$E_{i,n} = kG_n e_{(V_n \rtimes \cdots \rtimes V_{i+1}) \rtimes (D_{i+1} \times \cdots \times D_n)} \otimes_{kG_i} - : A_i\text{-mod} \rightarrow A_n\text{-mod}$$

and  $F_{i,n} = e_{(V_n \rtimes \cdots \rtimes V_{i+1}) \rtimes (D_{i+1} \times \cdots \times D_n)} kG_n \otimes_{kG_n} - : A_n\text{-mod} \rightarrow A_i\text{-mod}$ .

These functors are canonically left and right adjoint. Furthermore, there are canonical isomorphisms  $E_{j,n} \circ E_{i,j} \xrightarrow{\sim} E_{i,n}$  and  $F_{i,j} \circ F_{j,n} \xrightarrow{\sim} F_{i,n}$  for  $i \leq j \leq n$ .

Let  $\mathcal{A} = \bigoplus_{n \geq 0} A_n\text{-mod}$ ,  $E = \bigoplus_{n \geq 0} E_{n,n+1}$  and  $F = \bigoplus_{n \geq 0} F_{n,n+1}$ .

We denote by  $T$  the endomorphism of  $E$  given on  $E_{n-2,n}$  by right multiplication by

$$\hat{T}_{n-1} = q e_{V_n V_{n-1} D_{n-1} D_n} (n-1, n) e_{V_n V_{n-1} D_{n-1} D_n}.$$

We denote by  $X$  the endomorphism of  $E^2$  given on  $E_{n-1,n}$  by right multiplication by

$$\hat{X}_n = q^{n-1} e_{V_n D_n} e_{{}^t V_n} e_{V_n D_n}.$$

**Lemma 7.14.** *We have*

$$(\mathbf{1}_E T) \circ (T \mathbf{1}_E) \circ (\mathbf{1}_E T) = (T \mathbf{1}_E) \circ (\mathbf{1}_E T) \circ (T \mathbf{1}_E), \quad (T + \mathbf{1}_{E^2}) \circ (T - q \mathbf{1}_{E^2}) = 0$$

and  $T \circ (\mathbf{1}_E X) \circ T = q X \mathbf{1}_E$ .

*Proof.* The first statements involving only  $T$ 's are the classical results of Iwahori.

Let  $U$  be the subgroup of  $G_n$  with diagonal coefficients 1 and whose off-diagonal coefficients vanish except the  $(n, n-1)$ -th. We have

$$\begin{aligned} \hat{T}_{n-1} \hat{X}_{n-1} \hat{T}_{n-1} &= q^n e_{V_n V_{n-1} D_{n-1} D_n} e_U (n-1, n) e_{{}^t V_{n-1}} (n-1, n) e_U e_{V_{n-1} V_n D_{n-1} D_n} \\ &= q^n e_{V_n V_{n-1} D_{n-1} D_n} e_{{}^t V_n} e_{V_n V_{n-1} D_{n-1} D_n} = q e_{V_{n-1} D_{n-1}} \hat{X}_n e_{V_{n-1} D_{n-1}} \end{aligned}$$

and this induces the same endomorphism of  $E_{n-2,n}$  as  $q \hat{X}_n$ .  $\square$

Lemma 7.14 shows that we have a morphism  $H_n(q) \rightarrow \mathrm{End}(E_{0,n}) = \mathrm{End}_{kG_n}(kG_n/B_n)$  which sends  $T_i$  to the endomorphism given by right multiplication by  $q e_{B_n}(i-1, i) e_{B_n}$  and  $X_1$  to the identity, where  $B_n$  is the subgroup of  $G_n$  of upper triangular matrices (cf § 5.2.2). The classical result of Iwahori states that the restriction of this morphism to  $H_n^f$  is an isomorphism. This gives us a surjective morphism  $p : H_n \rightarrow H_n^f$  whose restriction to  $H_n^f$  is the identity. Since  $X_1$

maps to 1 in  $\text{End}(E_{0,n})$  and the quotient of  $H_n$  by  $X_1 - 1$  is isomorphic to  $H_n^f$ , it follows that  $p$  is the canonical map  $H_n \rightarrow H_n^f$ . In particular, the image of  $X_i$  is (up to an affine transformation) a Jucys-Murphy element :

$$p(X_i) = q^{1-i} T_{i-1} \cdots T_1 T_1 \cdots T_{i-1} = 1 + q^{1-i} (q-1) (T_{(1,i)} + q T_{(2,i)} + \cdots + q^{i-2} T_{(i-1,i)}).$$

We put  $R_n = \text{Hom}_{kG_n}(kG_n e_{B_n}, -) = e_{B_n} kG_n \otimes_{kG_n} - : A_n\text{-mod} \rightarrow H_n^f\text{-mod}$ . The multiplication maps

$$e_{B_i} kG_i \otimes_{kG_i} e_{V_n \cdots V_{i+1} D_n \cdots D_{i+1}} kG_n \rightarrow e_{B_n} kG_n$$

and

$$e_{B_n} kG_n e_{B_n} \otimes_{e_{B_i} kG_i e_{B_i}} e_{B_i} kG_i \rightarrow e_{B_n} kG_n e_{V_n \cdots V_{i+1} D_n \cdots D_{i+1}}$$

are isomorphisms. They induce isomorphisms of functors

$$R_i F_{i,n} \xrightarrow{\sim} \text{Res}_{H_i^f}^{H_n^f} R_n \quad \text{and} \quad \text{Ind}_{H_i^f}^{H_n^f} R_i \xrightarrow{\sim} R_n E_{i,n}.$$

**Remark 7.15.** The constructions carried out here make sense more generally for finite groups with a BN-pair and for arbitrary standard parabolic subgroups, the transpose operation corresponding to passing from the unipotent radical of a parabolic subgroup to the unipotent radical of the opposite parabolic subgroup. This produces a very general kind of “Jucys-Murphy element” in Hecke algebras of finite Weyl groups. In type  $B$  or  $C$ , we should recover the usual Jucys-Murphy elements.

Given  $a \in k^\times$ , let  $E_a$  be the generalized  $a$ -eigenspace of  $X$  acting on  $E$ .

**Lemma 7.16.** *The action of  $[E_a]$  and  $[F_a]$  on  $\bigoplus_{n \geq 0} K_0(A_n\text{-mod})$  gives a representation of  $\mathfrak{sl}_2$ . Furthermore, the classes of simple objects are weight vectors.*

*Proof.* Let  $\mathcal{O}$  be a complete discrete valuation ring with field of fractions  $K$  and residue field  $k$ . We consider the setting above where  $k$  is replaced by  $K$ . The functor  $\text{Hom}_{KG_n}(KG_n e_{B_n}, -)$  induces an isomorphism from the Grothendieck group  $L_n$  of the category of unipotent representations of  $KG_n$  to the Grothendieck group of the category of representations of the Hecke algebra of type  $\mathfrak{S}_n$  with parameter  $q$  over  $K$ . This isomorphism is compatible with the actions of  $E_a$  and  $F_a$ . It follows from §7.2.2 that  $E_a$  and  $F_a$  give a representation of  $\mathfrak{sl}_2$  on  $\bigoplus_{n \geq 0} L_n$  and the class of a simple unipotent representation of  $KG_n$  is a weight vector. Now, the decomposition map  $L_n \rightarrow K_0(A_n)$  is an isomorphism [Jam, Theorem 16.7] and the result follows.  $\square$

So, we have constructed an  $\mathfrak{sl}_2$ -categorification on  $\bigoplus_{n \geq 0} A_n\text{-mod}$  and a morphism of  $\mathfrak{sl}_2$ -categorifications  $\bigoplus_{n \geq 0} A_n\text{-mod} \rightarrow \bigoplus_{n \geq 0} H_n^f\text{-mod}$ .

**Remark 7.17.** Note that we deduce from this that the blocks of  $A_n$  correspond to the blocks of  $H_n^f$ .

7.3.2. We assume here only that  $\ell \nmid q$ . Let  $\mathcal{O}$  be the ring of integers of a finite extension of  $\mathbf{Q}_\ell$  and  $k$  be the residue field of  $\mathcal{O}$ .

Let us recall [FoSri] that the  $\ell$ -blocks of  $\text{GL}_n(q)$  are parametrized by pairs  $((s), (B_1, \dots, B_r))$  where  $s$  is a conjugacy class of semi-simple  $\ell'$ -elements of  $\text{GL}_n(q)$  and  $B_i$  is a block of  $\mathcal{H}_{n_i}(q^{d_i})$ , where  $C_{\text{GL}_n(q)}(s) = \text{GL}_{n_1}(q^{d_1}) \times \cdots \times \text{GL}_{n_r}(q^{d_r})$ . Let  $w_i$  be the  $e_i$ -weight of the block  $B_i$ , where  $e_i$  is the multiplicative order of  $q^{d_i}$  in  $k^\times$ . We define the weight of the block as the family  $\{(w_i, d_i)\}_{1 \leq i \leq r}$ .

**Theorem 7.18.** *Let  $R = k$  or  $\mathcal{O}$ . Two blocks of general linear groups over  $R$  with same weights are splendidly Rickard equivalent.*

*Proof.* The results on the local block theory of symmetric groups generalize to unipotent blocks of general linear groups [Br, §3] and we conclude as in the proof of Theorem 7.2 that the Theorem holds for unipotent blocks.

By [BoRou2], a block of a general linear group is splendidly Rickard equivalent to a unipotent block of a product  $\mathrm{GL}_{n_1}(q^{d_1}) \times \cdots \times \mathrm{GL}_{n_r}(q^{d_r})$  ([BoRou1, Théorème B] already provides a complex with homology only in one degree inducing a Morita equivalence). Such a block is splendidly Rickard equivalent to the principal block of  $\mathrm{GL}_{e_1 w_1}(q^{d_1}) \times \cdots \times \mathrm{GL}_{e_r w_r}(q^{d_r})$  by the unipotent case of the Theorem.  $\square$

**Remark 7.19.** Assume  $l|(q-1)$ . Then,  $k\mathrm{GL}_n(q)$  has a unique unipotent block, the principal block. The number of simple modules for such a block is the number of partitions of  $n$ . Consequently, a unipotent block of  $\mathrm{GL}_n(q)$  is not derived equivalent to a unipotent block of  $\mathrm{GL}_m(q)$  when  $n \neq m$ .

**Theorem 7.20.** *Let  $A$  be a block of a general linear group  $G$  over  $R = k$  or  $\mathcal{O}$ , let  $D$  be a defect group and  $B$  the corresponding block of  $N_G(D)$ . If  $D$  is abelian, then  $A$  and  $B$  are splendidly Rickard equivalent.*

*Proof.* By the result of [BoRou2] stated above, we may assume that  $A$  is a unipotent block. Then we proceed as in the proof of Theorem 7.6, using the fact that there is a unipotent block of a general linear group with defect group isomorphic to  $D$  that is splendidly Morita equivalent to the principal block of  $R(\mathrm{GL}_e(q) \wr \mathfrak{S}_w)$  for some  $w \geq 0$ , where  $e$  is the order of  $q$  in  $k^\times$  [Pu2, Mi, Tu].  $\square$

#### 7.4. Category $\mathcal{O}$ .

7.4.1. We construct here  $\mathfrak{sl}_2$ -categorifications on category  $\mathcal{O}$  of  $\mathfrak{gl}_n$ . In particular we show that the weak  $\mathfrak{sl}_2$ -categorification on singular blocks given by Bernstein, Frenkel and Khovanov [BeFreKho] is an  $\mathfrak{sl}_2$ -categorification.

We denote by  $\mathfrak{h}$  the Cartan subalgebra of diagonal matrices and  $\mathfrak{n}$  the nilpotent algebra of strictly upper triangular matrices of the complex Lie algebra  $\mathfrak{g} = \mathfrak{gl}_n$ . We denote by  $\mathcal{O}$  the BGG category of finitely generated  $U(\mathfrak{g})$ -modules that are diagonalisable for  $\mathfrak{h}$  and locally nilpotent for  $U(\mathfrak{n})$ .

Let  $\{e_{ij}\}$  be the standard basis of  $\mathfrak{g}$ , and let  $\varepsilon_1, \dots, \varepsilon_n$  be the basis of  $\mathfrak{h}^*$  dual to  $e_{11}, \dots, e_{nn}$ . For each  $\lambda \in \mathfrak{h}^*$  we denote by  $\lambda_1, \dots, \lambda_n$  the coefficients of  $\lambda$  with respect to  $\varepsilon_1, \dots, \varepsilon_n$ . We write  $\lambda \rightarrow_a \mu$  if there exists  $j$  such that  $\lambda_j - j + 1 = a - 1$ ,  $\mu_j - j + 1 = a$  and  $\lambda_i = \mu_i$  for  $i \neq j$ .

For each  $\lambda \in \mathfrak{h}^*$  let  $M(\lambda)$  be the Verma module with highest weight  $\lambda$  and let  $L(\lambda)$  be its unique irreducible quotient. Recall that  $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\lambda$ , where  $\mathfrak{b}$  is the subalgebra of upper-triangular matrices and  $\mathbf{C}_\lambda$  is the one-dimensional  $\mathfrak{b}$ -module on which  $e_{ii}$  acts as multiplication by  $\lambda_i$ .

Let  $\Theta$  be the set of maximal ideals of the center  $Z$  of  $U(\mathfrak{g})$ . For each  $\theta \in \Theta$  denote by  $\mathcal{O}_\theta$  the full subcategory of  $\mathcal{O}$  consisting of modules annihilated by some power of  $\theta$ . The category  $\mathcal{O}$  splits as a direct sum of the subcategories  $\mathcal{O}_\theta$ . Let  $\mathrm{pr}_\theta : \mathcal{O} \rightarrow \mathcal{O}_\theta$  denote the projection onto  $\mathcal{O}_\theta$ . Each Verma module belongs to some  $\mathcal{O}_\theta$ , and  $M(\lambda)$  and  $M(\mu)$  belong to the same subcategory if and only if  $\lambda$  and  $\mu$  are in the same orbit in the dot action of the Weyl group of  $\mathfrak{g}$  on  $\mathfrak{h}^*$ ,

i.e., if and only if  $(\lambda_1, \lambda_2 - 1, \dots, \lambda_n - n + 1)$  and  $(\mu_1, \mu_2 - 1, \dots, \mu_n - n + 1)$  are in the same  $\mathfrak{S}_n$ -orbit. We write  $\theta \rightarrow_a \theta'$  if there exist  $\lambda, \mu \in \mathfrak{h}^*$  such that  $M(\lambda) \in \mathcal{O}_\theta$ ,  $M(\mu) \in \mathcal{O}_{\theta'}$  and  $\lambda \rightarrow_a \mu$ .

Let  $V$  be the natural  $n$ -dimensional representation of  $\mathfrak{g}$ . The functor  $V \otimes - : \mathcal{O} \rightarrow \mathcal{O}$  decomposes as a direct sum  $\bigoplus_{a \in \mathbf{C}} E_a$ , where

$$E_a = \bigoplus_{\substack{\theta, \theta' \in \Theta \\ \theta \rightarrow_a \theta'}} \text{pr}_{\theta'} \circ (V \otimes -) \circ \text{pr}_\theta.$$

Each summand  $E_a$  has a left and right adjoint

$$F_a = \bigoplus_{\substack{\theta, \theta' \in \Theta \\ \theta \rightarrow_a \theta'}} \text{pr}_\theta \circ (V^* \otimes -) \circ \text{pr}_{\theta'}.$$

Let  $\lambda \in \mathfrak{h}^*$ . We have  $V \otimes M(\lambda) = V \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\lambda) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V \otimes \mathbf{C}_\lambda)$ , and therefore  $V \otimes M(\lambda)$  has a filtration with quotients isomorphic to the modules  $M(\lambda + \varepsilon_i)$ ,  $i = 1, \dots, n$ . Similarly  $V^* \otimes M(\lambda)$  has a filtration with quotients isomorphic to the modules  $M(\lambda - \varepsilon_i)$ ,  $i = 1, \dots, n$ . It follows that

$$[E_a M(\lambda)] = \sum_{\substack{\mu \in \mathfrak{h}^* \\ \lambda \rightarrow_a \mu}} [M(\mu)], \quad [F_a M(\lambda)] = \sum_{\substack{\mu \in \mathfrak{h}^* \\ \mu \rightarrow_a \lambda}} [M(\mu)]$$

in  $K_0(\mathcal{O})$ . Hence

$$[E_a F_a M(\lambda)] - [F_a E_a M(\lambda)] = c_{\lambda, a} [M(\lambda)],$$

where  $c_{\lambda, a} = \#\{i \mid \lambda_i - i + 1 = a\} - \#\{i \mid \lambda_i - i + 1 = a - 1\}$ . Because the classes of Verma modules are a basis for  $K_0(\mathcal{O})$ , we deduce that for each  $a \in \mathbf{C}$  the functors  $E_a$  and  $F_a$  give a weak  $\mathfrak{sl}_2$ -categorification on  $\mathcal{O}$  in which the simple module  $L(\lambda)$  has weight  $c_{\lambda, a}$ .

7.4.2. Given  $M$  a  $\mathfrak{g}$ -module, we have an action map  $\mathfrak{g} \otimes M \rightarrow M$ . Let  $X_M \in \text{End}_{\mathfrak{g}}(V \otimes M)$  be the corresponding adjoint map. This defines an endomorphism  $X$  of the functor  $V \otimes -$ . We have  $X_M(v \otimes m) = \Omega(v \otimes m)$  where  $\Omega = \sum_{i,j=1}^n e_{ij} \otimes e_{ji} \in \mathfrak{g} \otimes \mathfrak{g}$ .

Define  $T_M \in \text{End}_{\mathfrak{g}}(V \otimes V \otimes M)$  by  $T_M(v \otimes v' \otimes m) = v' \otimes v \otimes m$ . This defines an endomorphism  $T$  of the functor  $V \otimes V \otimes -$ .

**Lemma 7.21.** *We have the following equality in  $\text{End}_{\mathfrak{g}}(V \otimes V \otimes M)$ :*

$$T_M \circ (\mathbf{1}_V \otimes X_M) = X_{V \otimes M} \circ T_M - \mathbf{1}_{V \otimes V \otimes M}.$$

*Proof.* We have

$$\begin{aligned} X_{V \otimes M} T_M(v \otimes v' \otimes m) &= \sum_{i,j=1}^n e_{ij} v' \otimes e_{ji}(v \otimes m) \\ &= \sum_{i,j=1}^n e_{ij} v' \otimes e_{ji} v \otimes m + \sum_{i,j=1}^n e_{ij} v' \otimes v \otimes e_{ji} m \\ &= v \otimes v' \otimes m + T_M(\mathbf{1}_V \otimes X_M)(v \otimes v' \otimes m). \end{aligned}$$

□

The lemma implies that for each  $l$  we can define a morphism  $H_l(1) \rightarrow \text{End}_{\mathfrak{g}}(V^{\otimes l} \otimes M)$  by  $T_i \mapsto \mathbf{1}_V^{\otimes l-i-1} \otimes T_{V^{\otimes i-1} \otimes M}$  and  $X_i \mapsto \mathbf{1}_V^{\otimes l-i} \otimes X_{V^{\otimes i-1} \otimes M}$ . Jon Brundan has pointed out to us that this coincides up to shift (cf Remark 5.14) with an action described by Arakawa and Suzuki [ArSu, §2.2].

7.4.3. We shall now show that  $X$  and  $T$  restrict to give endomorphisms of the functors  $E_a$  and  $E_a^2$  which define  $\mathfrak{sl}_2$ -categorifications on  $\mathcal{O}$ . In view of Lemma 7.21, it suffices to identify  $E_a$  as the generalised  $a$ -eigenspace of  $X$  acting on  $V \otimes -$ .

To this end we observe that  $\Omega = \frac{1}{2}(\delta(C) - C \otimes 1 - 1 \otimes C)$ , where  $C = \sum_{i,j=1}^n e_{ij}e_{ji} \in Z$  is the Casimir element and  $\delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$  is the comultiplication. Furthermore  $C = \sum_{i=1}^n e_{ii}^2 + \sum_{1 \leq i < j \leq n} (e_{ii} - e_{jj}) + \sum_{1 \leq i < j \leq n} e_{ji}e_{ij}$  acts on the Verma module  $M(\lambda)$  as multiplication by  $b_\lambda = \sum_{i=1}^n \lambda_i^2 + \sum_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)$ . It follows that  $\Omega$  stabilizes any  $\mathfrak{g}$ -submodule of  $V \otimes M(\lambda) = L(\varepsilon_1) \otimes M(\lambda)$  and that the induced action on any subquotient isomorphic to  $M(\lambda + \varepsilon_i)$  is as multiplication by  $\frac{1}{2}(b_{\lambda+\varepsilon_i} - b_{\varepsilon_1} - b_\lambda) = \lambda_i - i + 1$ . Since  $V \otimes M(\lambda) = \bigoplus_{a \in \mathbb{C}} E_a M(\lambda)$ , this identifies  $E_a M(\lambda)$  as the generalised  $a$ -eigenspace of  $X_{M(\lambda)}$ . We deduce that for any  $M \in \mathcal{O}$ , the generalized  $a$ -eigenspace of  $X_M$  is  $E_a M$ .

**Remark 7.22.** The canonical adjunction between  $V \otimes -$  and  $V^* \otimes -$  is given by the canonical maps  $\eta : \mathbf{C} \rightarrow V^* \otimes V$  and  $\varepsilon : V \otimes V^* \rightarrow \mathbf{C}, v \otimes \xi \mapsto \xi(v)$ . Let  $X_M \in \text{End}_{\mathfrak{g}}(V^* \otimes M)$  and  $T_M \in \text{End}_{\mathfrak{g}}(V^* \otimes V^* \otimes M)$  be the induced endomorphisms (cf §4.1.2). Then  $X_M(\varphi \otimes m) = (-\Omega - n)(\varphi \otimes m)$  and  $T_M(\varphi \otimes \varphi' \otimes m) = \varphi' \otimes \varphi \otimes m$ .

## 7.5. Rational representations.

7.5.1. The construction of  $\mathfrak{sl}_2$ -categorifications in §7.4 works, more or less in the same way, on the category  $G\text{-mod}$  of finite-dimensional rational representations of  $G = GL_n(k)$ , where  $k$  is an algebraically closed field of characteristic  $p > 0$ .

Denote by  $\mathcal{X}$  the character group of the subgroup of diagonal matrices in  $G$ . We identify  $\mathcal{X}$  with  $\mathbf{Z}^n$  via the isomorphism sending  $(\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$  to  $\lambda = \sum_i \lambda_i \varepsilon_i \in \mathcal{X}$ , where  $\varepsilon_i$  is defined by  $\varepsilon_i(\text{diag}(t_1, \dots, t_n)) = t_i$ . This identifies the set  $\mathcal{X}_+$  of dominant weights with  $\{(\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n\}$ . For each  $\lambda \in \mathcal{X}_+$ , let  $L(\lambda)$  be the unique simple  $G$ -module with highest weight  $\lambda$ .

Let  $B$  be the Borel subgroup of upper triangular matrices in  $G$ . For each  $\lambda \in \mathcal{X}$ , the cohomology groups  $H^i(\lambda)$  of the associated line bundle on  $G/B$  are objects of  $G\text{-mod}$ . The alternating sums  $\chi(\lambda) = \sum_{i \geq 0} \text{ch}(H^i(\lambda)) \in \mathbf{Z}[\mathcal{X}]$  span the image of the embedding  $\text{ch} : K_0(G\text{-mod}) \rightarrow \mathbf{Z}[\mathcal{X}]$ .

The Weyl group  $W = \mathfrak{S}_n$  of  $G$  acts on  $\mathcal{X} = \mathbf{Z}^n$  by place permutations. This extends to an action of the affine Weyl group  $W_p$  generated by  $W$  together with the translations by  $p\varepsilon_i - p\varepsilon_{i+1}, 1 \leq i \leq n-1$ . Let  $Y$  be the group of permutations of  $\mathbf{Z}$  generated by  $d, \sigma_0, \dots, \sigma_{p-1}$ , where  $md = m+1$  and

$$m\sigma_a = \begin{cases} m+1 & \text{if } m \equiv a-1 \pmod{p} \\ m-1 & \text{if } m \equiv a \pmod{p} \\ m & \text{otherwise.} \end{cases}$$

The action of  $W_p$  on  $\mathcal{X} = \mathbf{Z}^n$  commutes with the diagonal action of  $Y$ .

**Lemma 7.23.** *Two elements  $\lambda, \mu \in \mathcal{X}$  have the same stabilizer in  $W_p$  if and only if they are in the same  $Y$ -orbit.*

*Proof.* Both conditions are equivalent to the following: for all  $i, j$ , and  $r$ , we have  $\lambda_i - \lambda_j = pr$  if and only if  $\mu_i - \mu_j = pr$ .  $\square$

We shall use the corresponding ‘dot actions’ obtained by conjugating by the translation by  $\rho = (0, -1, \dots, -n+1) \in \mathcal{X}$ :

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \quad \lambda \cdot y = (\lambda + \rho)y - \rho.$$

Let  $\Theta$  be the set of orbits of the dot action of  $W_p$  on  $\mathcal{X}$ . For each  $\theta \in \Theta$ , let  $\mathcal{M}_\theta$  be the full subcategory of  $G$ -mod consisting of modules whose composition factors are all of the form  $L(\lambda)$  for  $\lambda \in \theta$ . The Linkage Principle [CaLu] implies that  $G$ -mod decomposes as a direct sum  $G\text{-mod} = \bigoplus_{\theta \in \Theta} \mathcal{M}_\theta$ . Let  $\text{pr}_\theta : G\text{-mod} \rightarrow G\text{-mod}$  denote the projection onto  $\mathcal{M}_\theta$ . Given  $\lambda, \mu \in \mathcal{X}$  and  $a \in 0, \dots, p-1$ , we write  $\lambda \rightarrow_a \mu$  if there exists  $j$  such that  $(\lambda_j - j + 1) + 1 = \mu_j - j + 1 \equiv a \pmod{p}$  and  $\lambda_i = \mu_i$  for  $i \neq j$ . Note that  $\lambda \rightarrow_a \mu$  implies that  $w \cdot \lambda \rightarrow_a w \cdot \mu$  for all  $w \in W_p$ . For  $\theta, \theta' \in \Theta$ , we write  $\theta \rightarrow_a \theta'$  if there exist  $\lambda \in \theta$  and  $\mu \in \theta'$  such that  $\lambda \rightarrow_a \mu$ .

Let  $V$  be the natural  $n$ -dimensional representation of  $G$ . The left and right adjoint functors  $V \otimes - : G\text{-mod} \rightarrow G\text{-mod}$  and  $V^* \otimes - : G\text{-mod} \rightarrow G\text{-mod}$  decompose as direct sums  $\bigoplus_{0 \leq a \leq p-1} E_a$  and  $\bigoplus_{0 \leq a \leq p-1} F_a$ , where  $E_a$  and  $F_a$  are sums of translation functors, defined in the same way as in §7.4. The functors  $E_a$  and  $F_a$  have been studied extensively by Brundan and Kleshchev [BrKl].

Let  $e_a$  and  $f_a$  be the maps on characters induced by  $E_a$  and  $F_a$ . For each  $\lambda \in \mathcal{X}$ , we have (eg using [Jan, Proposition 7.8])

$$(7) \quad e_a \chi(\lambda) = \sum_{\substack{\mu \in \mathcal{X} \\ \lambda \rightarrow_a \mu}} \chi(\mu), \quad f_a \chi(\lambda) = \sum_{\substack{\mu \in \mathcal{X} \\ \mu \rightarrow_a \lambda}} \chi(\mu)$$

in  $\mathbf{Z}[\mathcal{X}]$ . Hence

$$e_a f_a \chi(\lambda) - f_a e_a \chi(\lambda) = c_{\lambda, a} \chi(\lambda),$$

where  $c_{\lambda, a} = \#\{i \mid \lambda_i - i + 1 \equiv a \pmod{p}\} - \#\{i \mid \lambda_i - i + 1 \equiv a - 1 \pmod{p}\}$ . We deduce that for each  $a \in \{0, \dots, p-1\}$  the functors  $E_a$  and  $F_a$  give a weak  $\mathfrak{sl}_2$ -categorification in which the simple module  $L(\lambda)$  has weight  $c_{\lambda, a}$ .

**7.5.2.** These weak  $\mathfrak{sl}_2$ -categorifications can be improved to  $\mathfrak{sl}_2$ -categorifications using the same procedure as in the characteristic zero case §7.4. We first define endomorphisms  $X$  of  $V \otimes -$  and  $T$  of  $V \otimes V \otimes -$ . Note that to define  $X$ , we first pass from  $G$ -modules to modules over  $\text{Lie}(G) = \mathfrak{gl}_n(k)$ . One small modification to the argument is required when  $p=2$ : in order to identify  $E_a$  with the generalized  $a$ -eigenspace of  $X$ , we write  $\Omega = -\delta(Z_2) + 1 \otimes Z_2 + Z_2 \otimes 1 + Z_1 \otimes Z_1 - \frac{n(n+1)}{2}$ , where  $Z_1 = \sum_{1 \leq i \leq n} e_{ii}$  and  $Z_2 = \sum_{1 \leq i < j \leq n} (e_{ii} - i)(e_{jj} - j) - \sum_{1 \leq i < j \leq n} e_{ji} e_{ij}$  are central elements of  $\text{Dist}(G)$  (cf [CaLu, §2.2]).

By composing the derived (and homotopy) equivalences arising from these  $\mathfrak{sl}_2$ -categorifications on  $G\text{-mod}$ , we obtain many equivalences.



**Theorem 7.24.** *Let  $\lambda$  and  $\mu$  be any two weights in  $\mathcal{X}$  with the same stabilizer under the dot action of  $W_p$ . Then there are equivalences*

$$K^b(\mathcal{M}_{W_p \cdot \lambda}) \xrightarrow{\sim} K^b(\mathcal{M}_{W_p \cdot \mu}) \quad \text{and} \quad D^b(\mathcal{M}_{W_p \cdot \lambda}) \xrightarrow{\sim} D^b(\mathcal{M}_{W_p \cdot \mu})$$

that induce the map

$$\chi(w \cdot \lambda) \mapsto \chi(w \cdot \mu)$$

on characters.

**Remark 7.25.** Rickard conjectured the existence of such equivalences for any connected reductive group having a simply-connected derived subgroup and whose root system has Coxeter number  $h < p$  [Ri2, Conjecture 4.1]. He proved the truth of his conjecture in the case of trivial stabilizers (under the weaker assumption  $h \leq p$ ). We do not place any restriction on  $p$  in Theorem 7.24.

*Proof.* By Lemma 7.23 we may assume that  $\mu = \lambda \cdot y$  where  $y \in \{d, \sigma_0, \dots, \sigma_{p-1}\}$ . If  $\mu = \lambda \cdot d$ , then we have an equivalence  $L(1, \dots, 1) \otimes - : \mathcal{M}_{W_p \cdot \lambda} \xrightarrow{\sim} \mathcal{M}_{W_p \cdot \mu}$ , given by tensoring with the determinant representation, that induces the desired map on characters.

Suppose that  $\mu = \lambda \cdot \sigma_a$ . Using the  $\mathfrak{sl}_2$ -categorification on  $G\text{-mod}$  provided by  $E = E_a$  and  $F = F_a$ , we obtain a self-equivalence  $\Theta$  of  $K^b(G\text{-mod})$  and of  $D^b(G\text{-mod})$  such that  $[\Theta] = s$  (Theorem 6.4). We define an  $\mathfrak{sl}_2$ -module  $U = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}u_i$  by  $eu_i = u_{i+1}$  for  $i \equiv a-1 \pmod{p}$  and  $eu_i = 0$  otherwise, and  $fu_i = u_{i-1}$  for  $i \equiv a \pmod{p}$  and  $fu_i = 0$  otherwise. Then  $su_i = u_{i+1}$  if  $i \equiv a-1 \pmod{p}$ ,  $su_i = -u_{i-1}$  if  $i \equiv a \pmod{p}$ , and  $su_i = u_i$  otherwise. Thus on the tensor power  $U^{\otimes n}$  we have  $su_\nu = (-1)^{h_-(\nu)} u_{\nu \sigma_a}$ , where  $u_\nu = u_{\nu_1} \otimes \dots \otimes u_{\nu_n}$  and  $h_-(\nu) = \#\{i \mid \nu_i \equiv a \pmod{p}\}$ .

By (7) we have a homomorphism of  $\mathfrak{sl}_2$ -modules  $U^{\otimes n} \rightarrow K_0(G\text{-mod})$ ,  $u_{\nu+\rho} \mapsto \chi(\nu)$ . It follows that  $s\chi(\nu) = (-1)^{h_-(\nu+\rho)} \chi(\nu \cdot \sigma_a)$ . Hence  $s\chi(w \cdot \lambda) = (-1)^{h_-} \chi(w \cdot \mu)$ , where  $h_- = h_-(w \cdot \lambda + \rho) = h_-(\lambda + \rho)$ . We conclude that  $\Theta[-h_-]$  restricts to equivalences  $K^b(\mathcal{M}_{W_p \cdot \lambda}) \xrightarrow{\sim} K^b(\mathcal{M}_{W_p \cdot \mu})$  and  $D^b(\mathcal{M}_{W_p \cdot \lambda}) \xrightarrow{\sim} D^b(\mathcal{M}_{W_p \cdot \mu})$  that induce the desired map on characters.  $\square$

**7.6.  $q$ -Schur algebras.** We explain in this part how to obtain  $\mathfrak{sl}_2$ -categorifications, and hence derived equivalences, for  $q$ -Schur algebras.

Let  $q \in k^\times$ . Let  $Y_n = \bigoplus_\lambda \text{Ind}_{H_\lambda^f}^{H_n^f} k$ , where  $\lambda = (\lambda_1 \geq \dots \geq \lambda_r)$  runs over the partitions of  $n$  and  $H_\lambda^f = \bigoplus_{w \in \mathfrak{S}_{[1, \lambda_1]} \times \dots \times \mathfrak{S}_{[n-\lambda_r+1, n]}} T_w k$  is the corresponding parabolic subalgebra of  $H_n^f$  and  $k$  corresponds to the representation 1. We define the  $q$ -Schur algebra  $S_n = \text{End}_{H_n^f}(Y_n)$ .

Let  $\mathcal{Y}_n$  be the full subcategory of  $H_n^f\text{-mod}$  whose objects are direct sums of direct summands of  $Y_n$  (“ $q$ -Young modules”) and let  $\mathcal{Y} = \bigoplus_{n \geq 0} \mathcal{Y}_n$ . Mackey’s formula shows that  $\mathcal{Y}$  is stable under  $E$  and  $F$ . For each of the  $\mathfrak{sl}_2$ -categorifications on  $\bigoplus_{n \geq 0} H_n^f\text{-mod}$  constructed in §7.2 we deduce from Theorem 5.32 an  $\mathfrak{sl}_2$ -categorification on  $\bigoplus_{n \geq 0} S_n\text{-mod}$  and a morphism of  $\mathfrak{sl}_2$ -categorifications  $\bigoplus_{n \geq 0} S_n\text{-mod} \rightarrow \bigoplus_{n \geq 0} H_n^f\text{-mod}$ . This provides a version of Theorem 7.12 for  $q$ -Schur algebras.

**Remark 7.26.** We go back to the setting of §7.3 (in particular,  $q$  is a prime power). The canonical map  $A_n \rightarrow \text{End}_{H_n^f \text{opp}}(kG_n e_{B_n})^{\text{opp}}$  is surjective and its image  $S'_n$  is Morita equivalent to  $S_n$  (“double centralizer Theorem”, cf [Ta]). This gives by restriction a fully faithful functor  $S_n\text{-mod} \xrightarrow{\sim} S'_n\text{-mod} \rightarrow A_n\text{-mod}$ . Since  $E(kG_n e_{B_n}) \simeq kG_{n+1} e_{B_{n+1}}$ , it follows

that  $\bigoplus_{n \geq 0} S_n\text{-mod}$  is stable under  $E$ . Mackey's formula shows that it is also stable under  $F$ . This gives a morphism of weak  $\mathfrak{sl}_2$ -categorifications  $\bigoplus_{n \geq 0} S_n\text{-mod} \rightarrow \bigoplus_{n \geq 0} A_n\text{-mod}$  and the composition with the morphism  $\bigoplus_{n \geq 0} A_n\text{-mod} \rightarrow \bigoplus_{n \geq 0} H_n^f\text{-mod}$  of §7.3.1 is isomorphic to the morphism  $\bigoplus_{n \geq 0} S_n\text{-mod} \rightarrow \bigoplus_{n \geq 0} H_n^f\text{-mod}$  constructed above. One deduces that  $\bigoplus_{n \geq 0} S_n\text{-mod} \rightarrow \bigoplus_{n \geq 0} A_n\text{-mod}$  is actually a morphism of  $\mathfrak{sl}_2$ -categorifications.

Note also that we get another proof of Lemma 7.16 using the fact that the canonical map  $K_0(S_n\text{-mod}) \xrightarrow{\sim} K_0(A_n\text{-mod})$  is an isomorphism.

**Remark 7.27.** The interested reader will extend the results of §7.5 to the quantum case and show that the categorification of  $q$ -Schur algebras can be realized as a subcategorification of the quantum group case.

## 7.7. Realizations of minimal categorifications.

7.7.1. We now show that the minimal categorification of §5.3 is a special case of the categorification on representations of blocks of cyclotomic Hecke algebras.

Fix  $a \in k^\times$  and put  $v = (v_1, \dots, v_n) = (a, \dots, a)$ . Then  $\mathcal{H}_i = \mathcal{H}_i(q, v)$  is the quotient of  $H_i$  by the ideal generated by  $x_1^n$  (where  $x_1 = X_1 - a$ ). The kernel of the action of  $H_i$  on the simple module  $K_i = H_i \otimes_{P_i} P_i/\mathfrak{m}_i$  contains  $x_1^n$  if and only if  $i \leq n$  (cf §3.2.1); let  $A_i$  be the block of  $\mathcal{H}_i$  containing  $K_i$  for  $0 \leq i \leq n$ . A finitely generated  $\mathcal{H}_i$ -module  $M$  is in  $A_i$  if and only if  $\mathfrak{n}_i$  acts nilpotently on  $M$  (equivalently  $\mathfrak{m}_i$  acts nilpotently on  $M$ ), and  $K_i$  is the unique simple module in  $A_i$ .

We have  $FM = 0$  for  $M \in A_0\text{-mod}$  and  $FM = \text{Res}_{\mathcal{H}_{i-1}}^{\mathcal{H}_i} M \in A_{i-1}\text{-mod}$  for  $M \in A_i\text{-mod}$  and  $0 < i \leq n$ .

Likewise  $EM = 0$  for  $M \in A_n\text{-mod}$ . Let  $M \in A_i\text{-mod}$  with  $0 \leq i < n$ . Consider  $N$  a simple  $\mathcal{H}_{i+1}$ -quotient of  $EM$ . We have  $\text{Hom}(EM, N) \simeq \text{Hom}(M, FN) \neq 0$ . In particular,  $FN$  has a non-zero  $\mathcal{H}_i$ -submodule  $M'$  on which  $x_1, \dots, x_i$  act nilpotently. Let  $M''$  be the  $(k[x_{i+1}] \otimes \mathcal{H}_i)$ -submodule of  $FN$  generated by  $M'$ . Then,  $x_1, \dots, x_{i+1}$  act nilpotently on  $M''$ . Now,  $N$  is a simple  $\mathcal{H}_{i+1}$ -module, hence it is generated by  $M''$  as a  $\mathcal{H}_{i+1}$ -module, so  $x_1, \dots, x_{i+1}$  act nilpotently on  $N$ . We deduce that they act nilpotently on  $EM$  as well. So,  $EM \in A_{i+1}\text{-mod}$ .

So  $\mathcal{A} = \bigoplus_i A_i\text{-mod}$  is an  $\mathfrak{sl}_2$ -categorification and  $\mathbf{Q} \otimes K_0(\mathcal{A})$  is a simple  $\mathfrak{sl}_2$ -module of dimension  $n+1$ . Let  $U = K_0 = k$ , the simple (projective) module for  $A_0 = k$ . The morphism of  $\mathfrak{sl}_2$ -categorifications  $R_U : \mathcal{A}(n) \rightarrow \mathcal{A}$  is an equivalence (Proposition 5.26). In particular  $\bar{H}_{i,n}$  and  $A_i$  are isomorphic, as each has an  $i!$ -dimensional simple module.

7.7.2. We explained in §3.3.2 that  $\bar{H}_{i,n}$  is Morita equivalent to its center, which is isomorphic to the cohomology of certain Grasmannian varieties. We sketch here a realization of the minimal categorification in that setting. We consider only the case  $q = 1$ ; the case  $q \neq 1$  can be dealt with similarly, replacing cohomology by  $\mathbf{G}_m$ -equivariant  $K$ -theory.

Let  $G_{i,j}$  be the variety of pairs  $(V_1, V_2)$  of subspaces of  $\mathbf{C}^n$  with  $V_1 \subset V_2$ ,  $\dim V_1 = i$  and  $\dim V_2 = j$ . We put  $A_i = H^*(G_i)$ . The  $(A_{i+1}, A_i)$ -bimodule  $H^*(G_{i,i+1})$  defines by tensor product a functor  $E_i : A_i\text{-mod} \rightarrow A_{i+1}\text{-mod}$  and switching sides, a left and right adjoint  $F_i : A_{i+1}\text{-mod} \rightarrow A_i\text{-mod}$ . Let  $E = \bigoplus E_i$  and  $F = \bigoplus F_i$ . This gives a weak  $\mathfrak{sl}_2$ -categorification that has been considered by Khovanov as a way of categorifying irreducible  $\mathfrak{sl}_2$ -representations. It is a special case of the construction of irreducible finite dimensional representations of  $\mathfrak{sl}_n$  due to Ginzburg [Gi].

We denote by  $X$  the endomorphism of  $E$  given on  $H^*(G_{i,i+1})$  by cup product by  $c_1(L_{i+1})$ . We have a  $\mathbf{P}^1$ -fibration  $\pi : G_{i,i+1} \times_{G_{i+1}} G_{i+1,i+2} \rightarrow G_{i,i+2}$  given by first and last projection. It induces a structure of  $H^*(G_{i,i+2})$ -module on  $H^*(G_{i,i+1} \times_{G_{i+1}} G_{i+1,i+2}) = H^*(G_{i,i+1}) \otimes_{H^*(G_{i+1})} H^*(G_{i+1,i+2})$ . There is a unique endomorphism  $T$  of  $H^*(G_{i,i+2})$ -module on  $H^*(G_{i,i+1} \times_{G_{i+1}} G_{i+1,i+2})$  satisfying  $T(c_1(L_{i+1})) = c_1(L_{i+2}) - 1$ . This provides us with an endomorphism of  $E_{i+1}E_i$  and taking the sum over all  $i$ , we get an endomorphism  $T$  of  $E^2$ . One checks easily that this gives an  $\mathfrak{sl}_2$ -categorification (with  $a = 0$ ) that is isomorphic to the minimal categorification.

The functor  $E^{(1,r)} : A_i\text{-mod} \rightarrow A_{i+r}\text{-mod}$  is isomorphic to the functor given by the bimodule  $H^*(G_{i,i+r})$ .

Take  $i \leq n/2$  and let us now consider  $\Theta[-i]$ , restricted to a functor  $D^b(H^*(G_i)\text{-mod}) \xrightarrow{\sim} D^b(H^*(G_{n-i})\text{-mod})$ . It is probably isomorphic to the functor given by the cohomology of the subvariety  $\{(V, V') | V \cap V' = 0\}$  of  $G_i \times G_{n-i}$ , the usual kernel for the Grassmannian duality (cf eg [KaScha, Exercice III.15]).

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